On the Birth and Growth of Pendulum Clocks in the Early Modern Era

Francesco Sorge, Marco Cammalleri and Giuseppe Genchi

Abstract Measuring the passage of time has intrigued humankind throughout the centuries. Ancient times witnessed the appearance and development of clepsydras and water clocks, whose place was subsequently taken by mechanical clocks in the Middle Ages. It is really surprising how the general architecture of mechanical clocks has remained almost unchanged in practice up to the present time. Yet the foremost mechanical developments in clock-making date from the 17th century, when the discovery of the laws of pendular isochronism by Galilei and Huygens permitted a higher degree of accuracy in the measuring of time.

1 The Art of Clock-Making Throughout the Centuries

1.1 Ancient Times: The Egyptian, Greek and Roman Ages

The first elements of temporal and spatial cognition among primitive societies were associated with the course of natural events. In practice, the starry heaven played the role of mankind's first huge clock. According to the philosopher Macrobius (4th century), even the Latin term *hora* derives, through the Greek word *'ώρα*, from an Egyptian hieroglyph pronounced *Heru* or *Horu*, which was Latinized into Horus and was the name of the Egyptian deity of the sun and the sky, the son of Osiris who was often represented as a hawk, the prince of the sky (Fig. 1).

Later on, the measure of time began to assume a rudimentary technical connotation and to benefit from the use of more or less ingenious devices. Various

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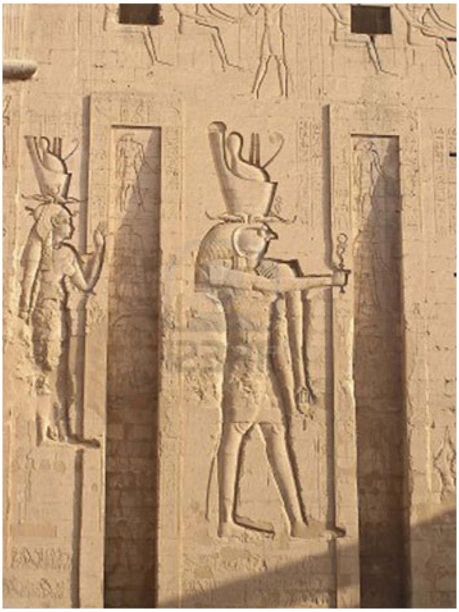


Fig. 1 The Egyptian deity Horu

kinds of clocks were developed with relatively high levels of accuracy by the Egyptian, Assyrian, Greek and Roman civilizations.

Starting from the well-known water clock of Ktesibios (Fig. 2), for which the flow rate into the measuring vessel was constant due to the constancy of the level h, an incredible degree of precision had been reached in Rome during the late imperial age with clepsydras (whose etymology is linked to the Greek words *κλέπτειν* + *'ύδωρ* = steal + water), clepsamias (from *κλέπτειν* + *'άμμος* = steal + sand), sundials or

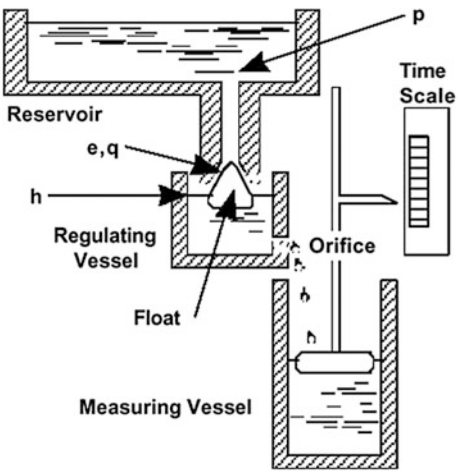


Fig. 2 Ktesibios' water clock (Alexandria, 3rd century BC)

sciateras (from *σκιή* + *θεωρεῖν* = shade + observe), and astrolabes (from the Latin words *astrum* + *labi* = star + slide).

Clepsydras were largely used in antiquity to measure short lapses of time, e.g., the peroration time in the courts of law. The Greek orator Lysias frequently suspended his speeches with the request, "*καί μοι 'επίλαβε το 'ύδωρ*" ("Please, stop the water for me"), so as to let his witnesses testify in front of the judges with no time constraint from the clepsydra (e.g., see speech *Against Pancleon*).

1.2 The Middle Ages

Some centuries later, the ancient clepsydras evolved into new types of water clocks and partly mechanical clepsydras, such as the little "alarm clocks" that were used in some monasteries of the Middle Ages consisting of a container that, once filled with water, let fall a metallic ball whose din awakened the provost.

It is not out of place to observe here that a constant speed for lowering of the water level can be theoretically obtained by a fourth degree parabolic shape of the meridian section of the container. In fact, the discharge velocity is proportional to the square root of the water level, while the flow rate is proportional to the product of the square of the container radius by the speed at which the level is lowered. Therefore, in order to obtain a constant lowering speed, the level must be proportional to the fourth power of the radius (see Fig. 3) and this shape seems to have been heuristically sought for some clepsydras of those days.

The first mechanical clocks appeared in the Byzantine and Islamic worlds, for both fixed and portable use, and in the subsequent early centuries of the Middle Ages, various types of weight clocks were owned by several notables and were generally equipped with a verge-and-foliot escapement (Fig. 4). At the same time, widespread construction of mechanical clocks of various sizes emerged, from big tower clocks to small pocket watches (Nuremberg eggs).

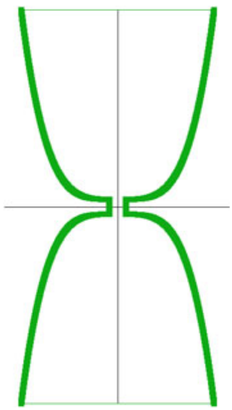


Fig. 3 Fourth degree parabolic shape of meridian section

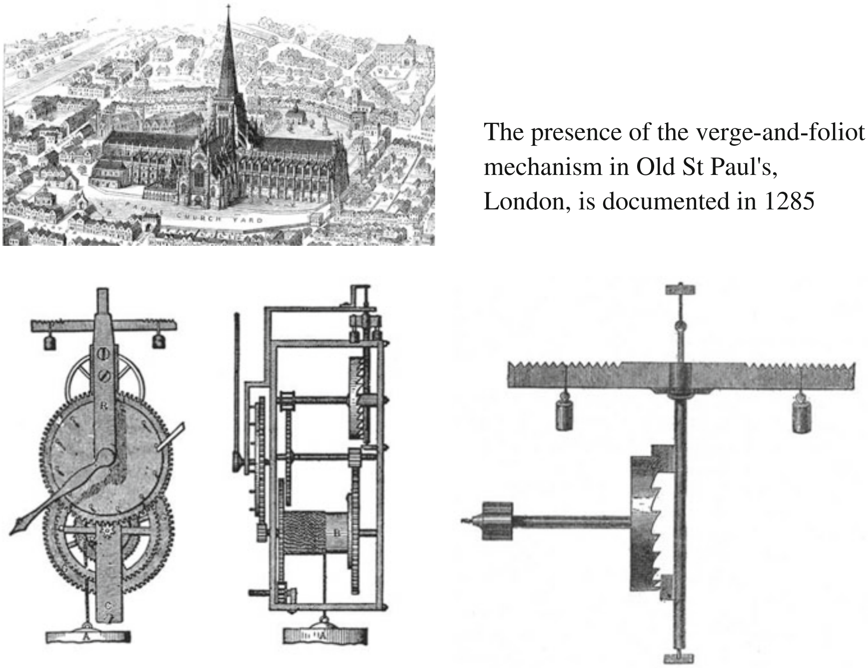
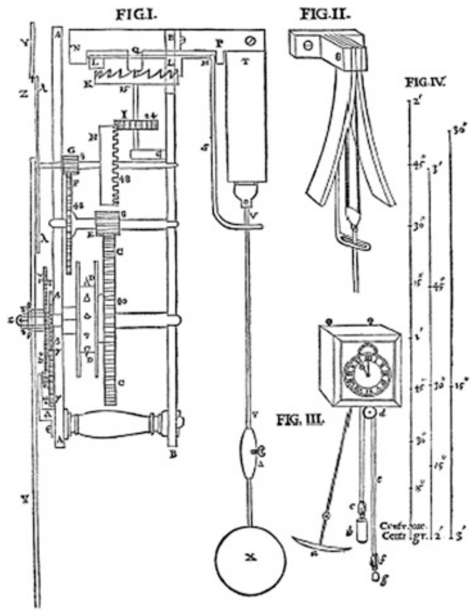


Fig. 4 Verge-and-foliot escapement. From "*Encyclopédie, ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers*", edited by Denis Diderot and Jean le Rond d'Alembert in Paris, 1751–1772. The presence of the verge-and-foliot mechanism in Old St Paul's, London, is documented in 1285

In the medieval period, the start of the hour count was different in respective European countries, though always with the same daily division of 24 h. Italy and Bohemia adopted the "*hora italica*" and "*hora bohemica*", both from one sunset to the next, France used the "*hora gallica*", from midnight to midnight, and British countries used the "*hora britannica*", from one sunrise to the next.

This period saw modern clock mechanisms assume their structure gradually, a structure that has somehow been present in all successive clocks, right up to the present day, though with a great number of refinements and improvements. The so-called "main" mechanism comprises the driving motor, the gear transmission and the dial plate with the hands, while the "secondary" mechanisms include: the charging system, which restores the potential energy; the distribution system or escapement, which transforms the uniform motion generated by the motor into a periodic series of small progressive movements; and the regulation mechanism, the task of which is to ensure a constant oscillation period.

The foliot regulation system dates from before 1285 AD, in which year we learn about the presence of this type of device in Old St. Paul's in London (Fig. 4). The foliot, the etymon of which is probably linked to the old French verb *folier* (to play or dance foolishly), was a horizontal balance bar carrying two weights, which



Christian Huygens' Horologium Oscillatorium ; Part One. 1
Translated and annotated by Ian Bruce.

CHRISTIAN HUYGENS
OF ZULICHEM, SON OF CONSTANTINE.

THE PENDULUM CLOCK
OR
THE MOTION OF PENDULUMS
ADAPTED TO CLOCKS.

By Geometrical Demonstrations.

This book is divided into five parts, the parts of which are concerned with : -

1. A ***description*** of the OSCILLATING CLOCK.

2. The ***fall of weights***, and the ***motion of these along a cycloid***.

3. The ***size and evolution of the curve***.

4. The ***centre of oscillation or movement***.

5. ***The construction of another kind of clock*** is shown, in which the motion of the pendulum is in a circle, and Theorems concerned with ***centrifugal force***.

Sixteen years ago we published a little book, after having invented the pendulum clock. Truly, observing that as we have discovered many things in that time along the way leading to the completion of the work, it seems fitting that the individual discoveries should be set out here in this book. Indeed, not only are these discoveries pertinent to the perfection of the clock we have invented, and as such they can be considered the most important part of it, but also these observations are relevant for the fundamentals of all such clock mechanisms, which had earlier been in a very poor state. Indeed it is not in the nature of a simple pendulum to provide equal and reliable measurements of time, since the wide lateral excursions often made may be observed to be slower than more narrow ones; however, we have been led in a different direction by geometry, from which we have found a means of suspending the pendulum, with which we were previously unacquainted, and by giving close attention to a line with a certain curvature, the time of the swing can be chosen equal to some calculated value and is seen clearly in practise to

Fig. 5 Verge and pallet escapement of Huygens' pendulum clock. On the *right*: title page of Huygens' treatise "*Horologium Oscillatorium sive de motu pendulorum*", 1673, translated by Ian Bruce

oscillated and interacted with a crown wheel through two pallets out of phase (Fig. 4). As no restoring force was acting on the system, the periodicity was referred to the foliot's inertia, so that the time measurement was highly inaccurate (Diderot and d'Alembert 1751–1772) .

A successive adjustment of the foliot, at the beginning of the 16th century, consisted of the replacement of the balance weights with two elastic steel ribbons, thanks to Peter Henlein, who was a locksmith in Nuremberg. Nevertheless, the definitive evolution of the verge-and-foliot escapement associated the verge and pallet system with pendular regulation and permitted a fairly satisfactory precision, using, in particular, the cycloidal pendulum of Huygens' clock (Fig. 5).

1.3 From Galilei's Pendulum to the Modern Mechanical Clock Regulation

The laws of the pendulum were first studied by Galileo Galilei at the end of the 16th century and after that by Christiaan Huygens in the 17th century. There is an age-old diatribe about the precedence of Galilei or Huygens in realizing the first pendulum clock. According to legend, Galilei began to reflect upon pendulum motion in 1581, after observing the oscillations of a lamp suspended inside the

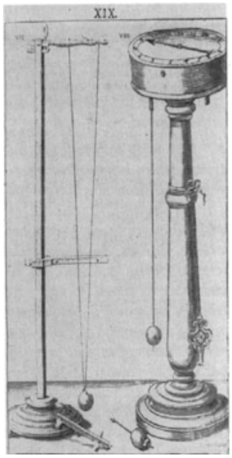


Fig. 6 Pendulums used by the *Accademia del Cimento* to measure oscillatory phenomena. The one on the left might be identified with Galilei's *pulsilogium*. From "*Saggi di naturali esperienze fatte nell'Accademia del Cimento"*, Florence, 1667

Cathedral of Pisa. He had the ingenious intuition that the oscillation period was somehow independent of the amplitude and conceived the functional dependence of the pendular period on the suspension length and the suspended weight.

The pendulum could be used as a tool to measure time intervals and, for example, could find an application in medicine in measuring pulse rate. Galilei had the idea of a "*pulsilogium*" in the last decade of 1500 (Fig. 6) and discussed it in Padua with his colleague Santorio, who described this medical device in two books of 1620 and 1622. The pendulum length was adjusted each time to synchronize the pulse frequency, thus permitting its calculation.

Many years later, in 1641, Galilei proposed the use of the pendulum as a regulatory mechanism for clocks and outlined the related design. However, he was now old and blind and did not accomplish that project. As a matter of fact, it is to be remarked that the ideal pendular motion is strictly isochronous only if the amplitude of its oscillations is very small, as was specified by Huygens a few decades after the first Galilean studies. Actually, the first pendulum clock was built in 1657 by Huygens, who also conceived the brilliant idea of the cycloidal trajectory, which ensures the theoretical isochronism even for large oscillation amplitudes (Huygens 1673).

A copy of the original design of Galilei's pendulum clock, which had been traced in those days by Vincenzo Viviani and Vincenzo Galilei, student and son of Galilei, respectively, is available to visitors of the Museum of Galilei in Florence (Fig. 7) and represents the device illustrated by their master in his letter of June 1637 to the Dutch admiral Laurens Reael in order to compete for a prize of 30,000 guldens. In this letter, he described his method for detecting the longitude offshore with the help of the so-called "Jovilabe", by comparing the local time with the hiding periods of Jupiter's Medicean satellites, Io, Europa, Ganymedes and Callisto. This comparison depended on the possibility of making an exact measurement of time, and to this end, Galilei proposed the idea for his own pendulum clock. Furthermore, Viviani also left a report on the process that led to the discovery of the pendulum laws and their possible application.

Fig. 7 Copy of the design of Galilei's clock mechanism by Vincenzo Viviani and Vincenzo Galilei. Copyright of *Museo Galileo*, photographic archives, Florence

Figure 8 shows a reconstruction of the pendulum clock with the Galilei escapement, which was realized in 1879 by the Florentine clock-maker Eustachio Porcellotti on the basis of Viviani's design and is preserved in the Museum of Galilei as well.

In spite of such previous studies by Galilei, the invention of the pendulum clock was claimed in 1658 by Huygens, whose primacy was hotly contested by Viviani.

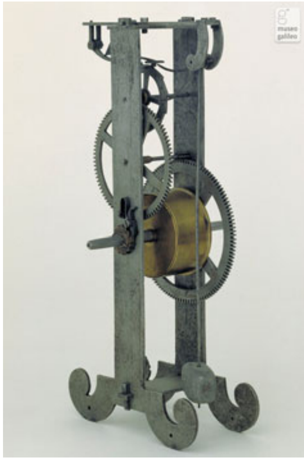


Fig. 8 Reconstruction of Galilei's pendulum clock by Eustachio Porcellotti (1879). Copyright of *Museo Galileo*, photographic archives, Florence

It is reported that, observing Viviani's designs, Huygens sharply declared: "It cannot work!".

Regulation by the balance-wheel-coil-spring system was later introduced by Hooke in the late 17th century, and in the meantime, the escapement evolved from the verge to the anchor, which was introduced by Clement in 1670, and then to the escapements of the deadbeat, cylinder and lever types, which were realized by Graham, Tompion and Mudge, respectively, in the 18th century, reaching a higher precision due to the elimination of any recoil movement (Fig. 9). Later on, the clock structure and the working technique would basically remain nearly unaltered throughout the modern and contemporary ages, until the recent appearance of electric clocks, which, however, did not cause the disappearance of mechanical clocks (Heidrick 2002).

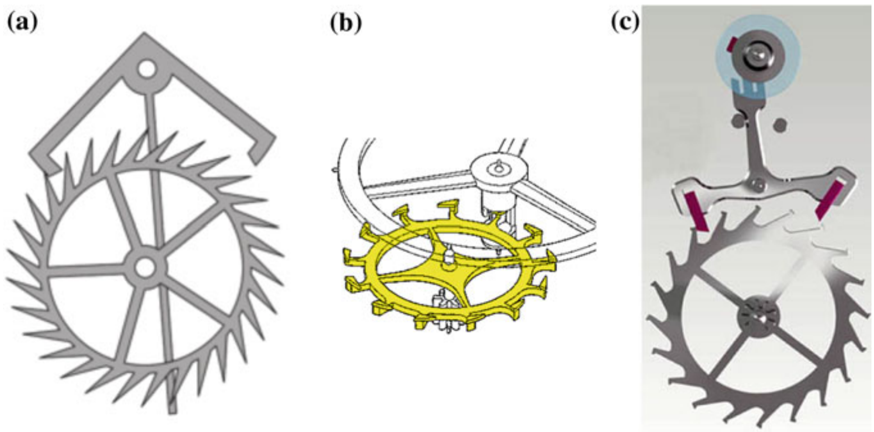


Fig. 9 Escapement of later centuries: **a** deadbeat escapement of Graham; **b** cylinder escapement of Tompion; **c** lever escapement of Mudge (Audemars Piguet)

2 Pendular Motion

2.1 The Pendulum Isochronism

Galilei described the pendular mechanism for clocks in great detail in 1641, but he did not accomplish that project owing to the infirmity of his age. Taking up a point discussed above, the ideal motion of the simple pendulum tends to become isochronous only if the amplitude of its oscillations is very small. When Huygens based his 1657 pendulum clock on the brilliant idea of cycloidal trajectory, the isochronism derived from the tautochronous property of the cycloid.

The study of the cycloid started with Galilei and continued with Fermat, Huygens, Newton and Bernoulli. Some relevant properties are:

- Indicating the radius of the generating circle with r, the evolute and the involute of a cycloid are two other identical cycloids, shifted a distance $2r$, upward and downward in the direction orthogonal to the base and a distance $r\pi$ in the direction parallel to it.
- The cycloid is tautochronous (*ταυτός χρόνος* = same time): a point mass always slides down to the bottom in the same lapse of time regardless of the starting position.
- The cycloid is brachistochronous (*βράχιστος χρόνος* = shortest time): the path that ensures the shortest sliding time from an upper fixed point to a lower one is an arc of cycloid, as may be proved by a variational approach using the Euler-Lagrange equation.

Observing Fig. 10 and using the notation reported there, since $ds_P = 2r\cos(\varphi/2)\, d\varphi$ according to the kinematical laws of rigid motion, the arc of cycloid measured

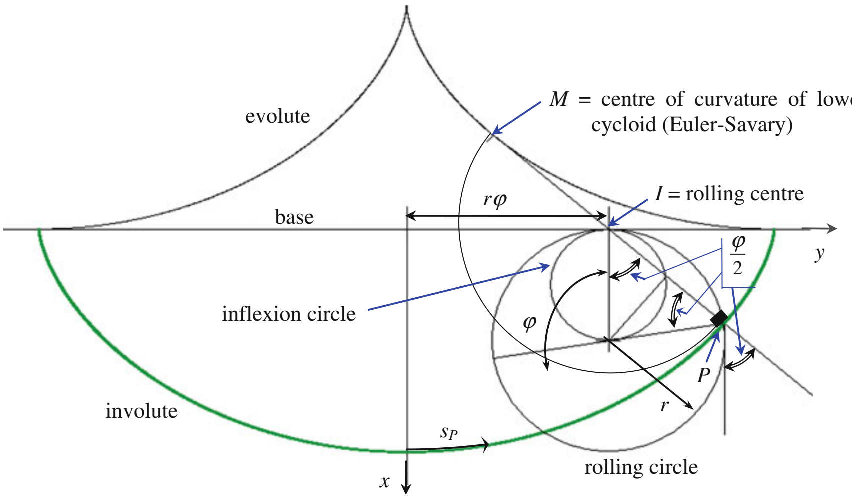


Fig. 10 Cycloid tautochronism

from the bottom may be written in the form $s_P = 4r\sin(\varphi/2)$, where r is the radius of the generating circle, φ is its rolling angle and $\varphi/2$ is also the local slope at P. Thus, the gravitational restoring force is proportional to s_P and the equilibrium of the point mass P along the path yields,

$$m\ddot{s}_P = -mg\sin\left(\frac{\varphi}{2}\right) \rightarrow \ddot{s}_P = -\frac{gs_P}{4r} \rightarrow \dot{s}_P^2 = -\frac{g}{4r}\left(s_P^2 - s_{P,\max.}^2\right), \tag{1}$$

whence, integrating again, the sliding time from the top position to the bottom turns out to be the same for any starting position:

$$\frac{T_{\text{sliding}}}{2}\sqrt{\frac{g}{r}} = \left|\sin^{-1}\left(\frac{s_P}{s_{P,\max.}}\right)\right|_0^{s_{P,\max.}} = \sin^{-1}(1) \rightarrow T_{\text{sliding}} = \pi\sqrt{\frac{r}{g}}. \tag{2}$$

The tautochronous property of the cycloidal path was experimentally proved by the Dutch scientist W.J. Gravesande, using the device shown in Fig. 11, which was described in his treatise “Physices Elementa Mathematica” and has been recently

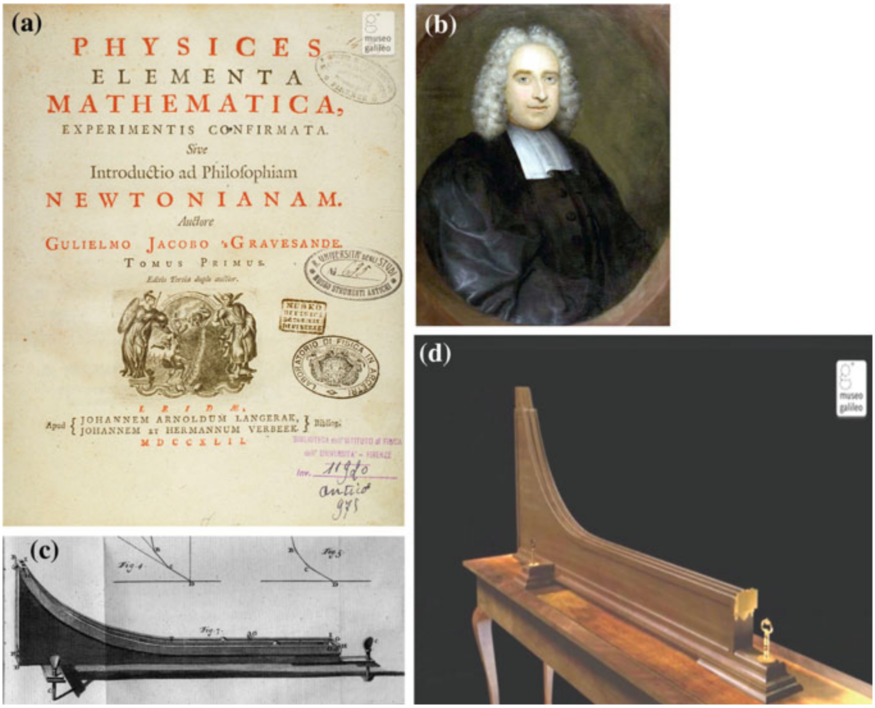


Fig. 11 **a** Gravesande’s treatise “Physices Elementa Mathematica”. **b** Willem Jacob’s Gravesande. **c** Device for experimental tests on the cycloid tautochronism (by Gravesande). **d** Reconstruction of Gravesande’s device by Museum Galileo, Florence. Copyright of *Museo Galileo*, photographic archives, Florence

reconstructed by the "Museo Galileo" in Florence. Letting two balls roll along two identical cycloidal tracks, starting from two different rest positions, they arrive together at the bottom, though they cross the finishing line with different velocities because of the law of the conservation of energy (see Eq. 1).

The tautochronous property of the cycloid is strictly associated with the isochronism of the cycloidal pendulum. Figure 12 shows that, when the flexible red ribbon OMP oscillates, wrapping and unwrapping the two rigid cycloidal bands generated by a circle of diameter $2r$, the point mass P, which is located at a distance $4r$ on the ribbon, describes a cycloidal trajectory equal to those bands, but shifted a distance $2r$ in the downward direction and symmetrically placed between them (involute of the upper rigid cycloids).

Actually, as $s_M = 4r - MP = 4r - MP_0 = 4r[1 - \sin(\pi/2 - \psi/2)]$ (see Fig. 10 and previous discussion on tautochronism), one has

$$\begin{aligned} x_P &= x_M + (4r - s_M)\cos(\psi/2) \\ &= r(1-\cos\psi) + 4r\cos^2(\psi/2) = 2r + r(1+\cos\psi), \\ y_P &= y_M + (4r - s_M)\sin(\psi/2) \\ &= r(\psi - \sin\psi) + 4r\sin(\psi/2)\cos(\psi/2) = r(\psi + \sin\psi), \end{aligned} \tag{3a, b}$$

which are just the parametric equations of the lower cycloid.

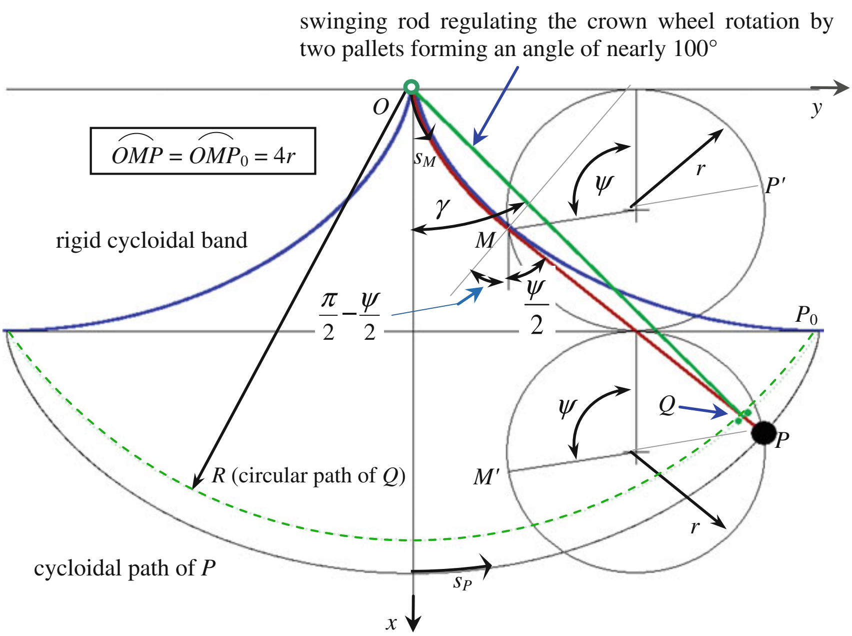


Fig. 12 Isochronism of the cycloidal pendulum

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The oscillation period is four times the time interval elapsed between the maximum amplitude position and the bottom position, and thus, the oscillations are isochronous due to the tautochronism properties of the cycloidal path.

The brachistochronous property of the cycloids is also interesting, though of minor concern for the pendular motion. The space covered by a point mass to slide from a fixed upper position to a fixed lower position along a generic path should be calculated by integrating the following expression:

$$\frac{ds}{dt} = \frac{dy}{dt}\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{2g(x_0 - x)}, \tag{4}$$

where the subscript 0 indicates the starting level. Hence, putting $x_0 - x = X$ and indicating with a prime the differentiation with respect to y, one has

$$\sqrt{2g} \times dt = dy \times \sqrt{\frac{1 + X'^2}{X}} = dy \times f(X, X') \quad \text{where} \quad f = \sqrt{\frac{1 + X'^2}{X}}. \tag{5a, b}$$

The condition that the sliding time to pass from level x_0 to level $x < x_0$ is the shortest one implies minimization of the integral of the right hand of Eq. (5a) from y_0 to y, whence the Euler-Lagrange equation is

$$\frac{df}{dX} - \frac{d}{dy}\left(\frac{df}{dX'}\right) = 0. \tag{6}$$

Replacing the function f by Eq. (5b), the solution of Eq. (6) can be found to be $X(1 + X'^2) = \text{constant} = 2r$, which gives $X' = \pm\sqrt{2r/X - 1}$. Hence one gets, integrating again,

$$\pm(y - y_0) = \sqrt{X(2r - X)} + 2r\tan^{-1}\sqrt{2r/X - 1}. \tag{7}$$

This is just the Cartesian equation of the cycloid passing through the point (x_0, y_0) and stemming from the rolling motion of a circle of radius r under a straight horizontal base line. Actually, replacing $X = r(1 + \cos\psi)$, $(y - y_0) = \pm r(\psi + \sin\psi)$ into Eq. (7), an identity is obtained.

2.2 The Structure of Galilei's Clock Mechanism. Ideal and Actual Operation

The ideal oscillation period of the simple pendulum may be calculated by well-known procedures ignoring the impulse supply and the energy dissipation in the whole clock mechanism. Defining the swing angle by θ (e.g., positive in the anticlockwise direction), introducing the dimensionless time variable $\tau = \omega_n t$, where

$\omega_n = \sqrt{g/l}$ and indicating the derivatives with respect to τ with primes, the motion equation shows the familiar trigonometric law of the restoring force

$$\theta'' + \sin\theta = 0. \tag{8}$$

The first integration gives

$$\frac{\theta'^2}{2} = 2\left(\sin^2\frac{\Theta}{2} - \sin^2\frac{\theta}{2}\right), \tag{9}$$

where Θ is the oscillation amplitude.

Hence, putting $\sin^2(\Theta/2) = k^2$ and $\sin^2(\theta/2) = k^2\sin^2 u$, the change of the variable from θ to u leads to a Legendre normal form, which permits calculating the dimensionless oscillation period T by the second integration

$$d\tau = \frac{du}{\sqrt{1 - k^2\sin^2 u}} \to T = 4K(k) = 4K\left(\sin\frac{\Theta}{2}\right), \tag{10a, b}$$

where $K(k)$ stands for the complete elliptic integral of the first kind with modulus k. This result reveals the dependence of the period on the swing amplitude and, as the complete elliptic integral $K(k)$ is equal to $\pi/2$ for $k = 0$ and is an increasing function of k, the period decreases monotonicly on decreasing the amplitude and approaches the harmonic period 2π for small oscillation widths.

The cycloidal pendulum described by Huygens in his treatise "*Horologium Oscillatorium sive de motu pendulorum*" in 1673 is not affected by this drawback, because it is based on the tautochronous property of the cycloidal trajectory, along which a point mass always slides down in the same lapse of time under the influence of gravity regardless of its starting position.

However, a deeper reflection is here advisable on the fact that, strictly speaking, all the above considerations on the ideal isochronism or non-isochronism of pendular motion are somewhat illusory in practice, due to the unavoidable friction losses present in the whole clock assembly and to the consequent impulse supply necessary to provide the dissipated energy periodically.

Figure 13 shows a mechanical reconstruction of Galilei's clock prototype on the basis of Viviani's design and clearly highlights the functionality of the Galilean escapement. The escapement wheel has ten ratchets on the crown and ten front pegs. Its intermittent motion is controlled by a catch, which is slightly loaded by a thin spring, and by two curved levers fastened to the pendulum hinge.

On approaching the left dead position of the pendulum near the oscillation end, for a certain angle $\theta_1 < 0$, the upper releasing lever raises the catch until the wheel is left free and rotates to contact the lower impulse lever with its peg (pendulum position θ_2). After a short recoil to reach the dead position, the wheel peg pushes the impulse lever until leaving it for $\theta = \theta_3$, near the right dead position, providing the energy lost by friction during the cycle.

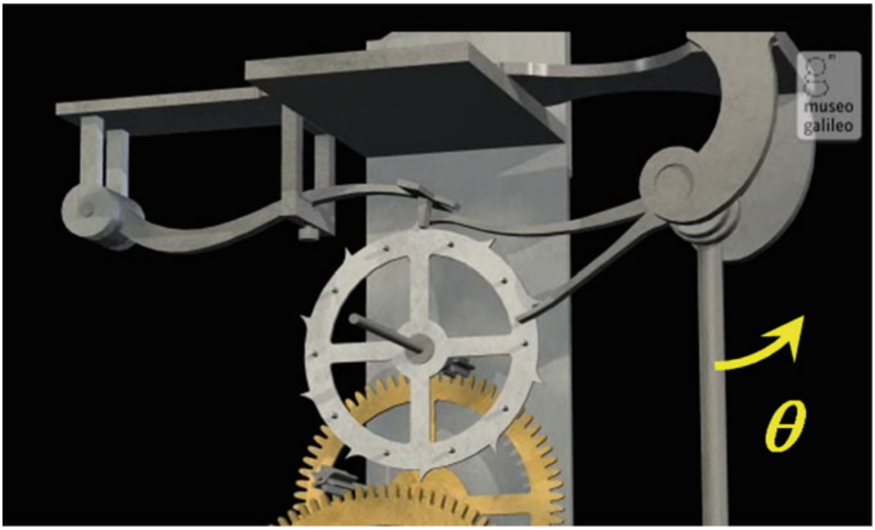

Fig. 13 Reconstruction of Galilei's escapement on the basis of the design of Vincenzo Viviani and Vincenzo Galilei. Copyright of *Museo Galileo*, photographic archives, Florence

Considering the actual operation of the pendulum machine, Eq. (8) changes into

$$\theta'' + \theta = \theta - \sin\theta + \frac{M_{\text{mot.}}(\theta)}{mgl} - \frac{M_{\text{hinge}}\text{sgn}(\theta')}{mgl} - \frac{M_{\text{air}}\text{sgn}(\theta')}{mgl}\omega_n^{\mu}|\theta'|^{\mu} + \frac{M_{\text{rel.}}(\theta)}{mgl}, \tag{11}$$

where m is the pendulum mass and the four moments M are defined as follows:

- $M_{\text{mot.}}$ is the motive torque acting on the impulse lever, which varies with the pendulum tilt, is active from the starting position θ_2 up to the final one θ_3, where the peg leaves the impulse lever, and is proportional to the mutual force between the peg and the lever. This force is in turn proportional to the driving torque M_0, applied to the escapement wheel by the motor weight through the whole gear train, and is also a function of θ and of the sliding direction of the peg, which determines the sign of the friction angle.
- M_{hinge} is the absolute value of the friction torque in the pendulum hinge.
- $M_{\text{air}}(\omega_n\theta')^{\mu}$ indicates the air resistance, which is supposed to be a function of the μth power of the oscillating velocity, where μ is >1.
- $M_{\text{rel.}}$ is the moment of the force necessary to release the ratchet, which is supposed active between the positions θ_1 and θ_2, of beginning and ending of the contact between the releasing lever and the catch (it is supposed that the peg contacts the impulse lever immediately after the wheel release).

The dissipative torque M_{hinge} in the pendulum hinge is piece-wise continuous and constant and changes its direction at the motion inversion, whereas the torque M_{air}, due to the air resistance, is continuous and depends on the oscillation speed. These two torques are both active during the whole period. The releasing torque $M_{\text{rel.}}$ may be assumed constant but is active only during a short fraction of the period. The moving torque $M_{\text{mot.}}$ is also active during a partial fraction of the oscillation period, from θ_2 to θ_3, and though the driving torque M_0 exerted on the escapement wheel by the gears may be plausibly considered constant, yet $M_{\text{mot.}}$ varies with the position on the impulse lever of the contact point with the peg and also depends on the direction of the sliding friction. The total force between the peg and the lever can be calculated by the rotational equilibrium condition of the escapement wheel and permits calculating $M_{\text{mot.}}$ by imposing the rotational equilibrium of the impulse lever. The relation between $M_{\text{mot.}}$ and M_0 is somewhat complex and is not reported. Here, it is only mentioned that this relation depends on the pendulum angle θ, on the radial position of the pegs, on their diameter, on the angle between the lever and the pendulum rod, on the offset of the lever with respect to the centre of the pendulum hinge and on the sliding direction.

Assuming one of the previous tilt angles θ as a small reference parameter ε, e.g., $\theta_1 = \varepsilon$, scaling all the angles θ by ε and minding the series expansion of the sine function, the difference $\theta - \sin\theta$ is of order ε^3. If one supposes that all four dimensionless moments of Eq. (11) exert an influence on the pendulum motion that is comparable with the gravitational nonlinearity, they must be regarded as of order ε^3 as well. Otherwise, some of them may be regarded as of a lower or higher order of magnitude, i.e., of order ε^n with $n \neq 3$.

The right hand of Eq. (11) is characterized by discontinuities of the first kind, but, looking only for a first order approximation of the solution, an averaging approach of the Krylov-Bogoliubov (K-B) type appears appropriate (Krylov and Bogoliubov 1947). Putting $\theta = \varepsilon\beta$, letting $\beta = B\sin(\tau + \phi)$ be the zero order solution (for $\varepsilon \rightarrow 0$) and indicating the right hand of Eq. (11) with $\varepsilon^3 F(\beta)$, the K-B procedure assumes that B and ϕ are not two constants but two functions of τ and eliminates the new degree of freedom that is being introduced by imposing the further condition $\beta' = B\cos(\tau + \phi)$, whence

$$\begin{aligned} B' \sin(\tau+\phi) + B\varphi' \cos(\tau+\phi) &= 0, \\ B' \cos(\tau+\phi) - B\phi' \sin(\tau+\phi) &= \varepsilon^2 F(\beta), \end{aligned} \tag{12a, b}$$

and consequently,

$$B' = \varepsilon^2 F(\beta)\cos(\tau+\phi), \quad B\phi' = -\varepsilon^2 F(\beta)\sin(\tau+\phi). \tag{13a, b}$$

Equations (13a, b) imply that B and ϕ are slowly varying functions of τ, and thus, they can be approximately averaged in the short period 2π neglecting their variation:

$$B' \cong \frac{\varepsilon^2}{2\pi}\int_0^{2\pi} F(\beta)\cos(\tau+\phi)d(\tau+\phi),$$
$$B\phi' \cong -\frac{\varepsilon^2}{2\pi}\int_0^{2\pi} F(\beta)\sin(\tau+\phi)d(\tau+\phi). \quad (14a, b)$$

Therefore, fixing the functional dependence on θ of the three dimensionless dissipative moments, $M_{\text{hinge}}/(mgl)$, $M_{\text{air}}(\omega_n\varepsilon)^{\mu}/(mgl)$ and $M_{\text{rel.}}/(mgl)$, which are contained in $F(\beta)$, and assuming steady oscillations ($B' = 0$), it is possible to solve for the required driving torque M_0 and for the frequency change $\omega_n\phi'$ depending on the oscillation amplitude $\Theta = \varepsilon B$.

Figure 14 illustrates these results with an example case and shows the difference between two choices of the order of magnitude of the four moments $M_{(\ldots)}$ in Eq. (11): $n = 2$ indicates that the nonlinear gravitational effect is of a lower order, whereas this effect is comparable with the dissipative and impulsive effects for $n = 3$. The red curve gives the theoretical dependence of the period on the amplitude and corresponds to an exponent $n \gg 3$. The exponent μ of the air resistance was fixed to the value $\mu = 1.5$, in the implicit hypothesis of an intermediate viscous-turbulent condition. What is most interesting in the results is that the overall effect of the driving impulse and the dissipation sources somehow counterbalances the period increase of the ideal pendulum in increasing the oscillatory amplitude and may even isochronize the motion in particular conditions under which the nonlinear effects are all comparable with each other.

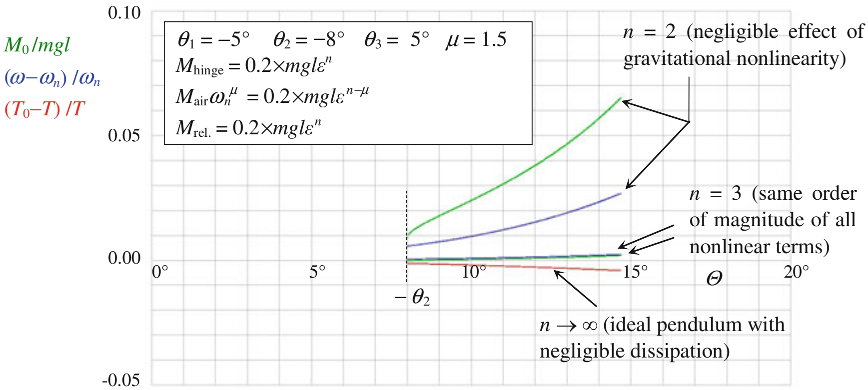

Fig. 14 Dimensionless driving torque on escapement wheel M_0/mgl (in *green*) and angular frequency relative change $(\omega - \omega_n)/\omega_n$ (in *blue*) versus oscillation amplitude Θ, for two values of the order n. Nonlinear period change of ideal pendulum (in *red*), $T_0 = 2\pi$, $T = 4K(k)$

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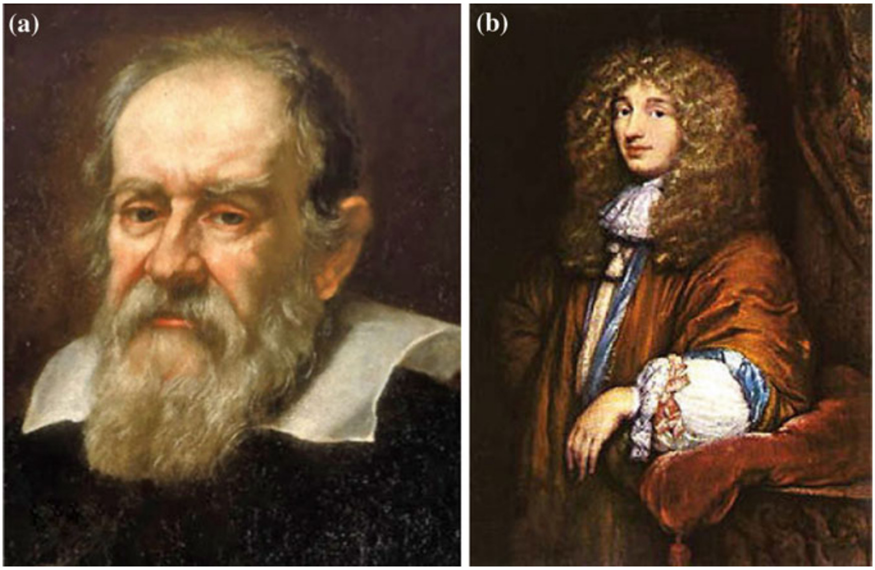

Fig. 15 **a** Galileo Galilei (Pisa 1564, Arcetri 1642); **b** Christiaan Huygens (The Hague 1629, The Hague 1695)

3 Conclusive Remarks

An animated debate arose in the 17th century between the Dutch scientist Christiaan Huygens and Galilei's heirs, Vincenzo Viviani and Vincenzo Galilei, about the primacy of the invention and construction of the first pendulum clock (Fig. 15). What we may conclude now is that both were to be considered fathers of this ingenious instrument: Galilei for studying the laws of pendular motion first, understanding their application in the measurement of time and conceiving the pendulum clock; Huygens for his successful discovery of the tautochronous property of the cycloidal trajectory, which permits attaining the theoretical pendulum isochronism, and for the actual construction of the first pendulum clock.

A careful analysis of the combined effects of the unavoidable dissipation sources present in the clock assembly and of the necessary periodic impulses to be provided in order to restore the lost energy highlights the slight deviation of the real operation from the ideal theory, the results of which then appear somewhat illusory, and suggests the possible isochronization of the simple pendulum motion under particular dissipation conditions.

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