On the isochronism of Galilei's *horologium*

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Abstract – Measuring the passage of time has always fascinated the humankind throughout the centuries. It is amazing how the general architecture of clocks has remained almost unchanged in practice to date from the Middle Ages. However, the foremost mechanical developments in clock-making date from the seventeenth century, when the discovery of the isochronism laws of pendular motion by Galilei and Huygens permitted a higher degree of accuracy in the time measure.

Keywords: Time Measure, Pendulum, Isochronism

Brief Survey on the Art of Clock-Making

The first elements of temporal and spatial cognition among the primitive societies were associated with the course of natural events. In practice, the starry heaven played the role of the first huge clock of mankind. According to the philosopher Macrobius (4th century), even the Latin term *hora* derives, through the Greek word *'ώρα*, from an Egyptian hieroglyph to be pronounced *Heru* or *Horu*, which was Latinized into Horus and was the name of the Egyptian deity of the sun and the sky, who was the son of Osiris and was often represented as a hawk, prince of the sky.

Later on, the measure of time began to assume a rudimentary technical connotation and to benefit from the use of more or less ingenious devices. Various kinds of clocks developed to relatively high levels of accuracy through the Egyptian, Assyrian, Greek and Roman civilizations. Starting from the well known water clock of Ktesibios, an incredible grade of precision had been reached in Rome during the late imperial age by clepsydras (whose etymology is linked to the Greek words *κλέπτειν* + *'ύδωρ* = steal + water), clepsamias (from *κλέπτειν* + *'άμμος* = steal + sand), sundials or sciateras (from *σκιή* + *θεωρεῖν* = shade + observe), astrolabes (from the Latin words *astrum* + *labi* = star + slide). Clepsydras were largely used in the antiquity to measure short lapses of time, e. g. the peroration time in the courts of law. The Greek orator Lysias frequently intersperses the request "*καί μοι 'επίλαβε το 'ύδωρ*" (= please, stop the water for me) in his speeches, to let his witnesses testify in front of the judges with no time constraint by the clepsydra.

Some centuries later, ancient clepsydras evolved towards new types of water clocks and mechanical clepsydras, e. g. like the little wakers that were used in some monasteries of the Middle Ages and consisted of a container that, once filled of water, let fall a metallic ball whose din awaked the provost. It is not out of place to observe here that a constant lowering speed of the water level can be theoretically obtained by a fourth degree parabolic shape of the meridian section of the container and this shape seemed to be heuristically sought by some clepsydras of those days.

The first mechanical clocks appeared in the Byzantine and Islamic worlds, for both the fixed and portable use. In the following early centuries of the Middle Ages, various types of weight clocks were owned by many notables, generally equipped with the verge-and-foliot escapement (Fig. 1), and many mechanical clocks were built in various sizes, from the big tower clocks to the small pocket watches (Nuremberg eggs). In those times, the start of the hour count was different in the various countries of Europe, though always with the same day division in 24 hours. Italy and Bohemia adopted the "*hora italica*" and "*hora bohemica*", both from one sunset to the next, France used the "*hora gallica*", from midnight to midnight, British countries used the "*hora britannica*", from the sun-up to the next one.

This period sees the modern clock mechanisms gradually assuming their structure, which is somehow present in all successive clocks, in practice till today, though with a great number of refinements and improvements. The so-called "main" mechanism includes the driving motor, the gear transmission and the dial plate with the hands, while the "secondary" mechanisms comprise: the charging system, which restores the potential energy; the distribution system or escapement, which transforms the uniform motion generated by the motor into a periodic series of small progressive movements; the regulation mechanism, whose task is to ensure a constant oscillation period.

The foliot regulation system dates from before 1285 A. D., in which year we learn about the presence of this type of device at Old St. Paul's in London. The foliot, whose etymon is probably linked to the old French verb *folier* = to play or dance foolishly, was a horizontal balance bar carrying two weights, which oscillated and interacted with a crown wheel through two pallets (Fig. 1). As no restoring force was acting on the system, the periodicity was demanded to the foliot inertia, so that the time measurement was highly inaccurate. A successive modification of the foliot, at the beginning of the 16th century, consisted in the replacement of the balance weights by elastic steel ribbons thanks to Peter Henlein who was a locksmith in Nuremberg, but the definitive evolution of the verge-and-foliot escapement associated the verge and pallet system with the pendular regulation and permitted a fairly satisfactory precision (Fig. 2).

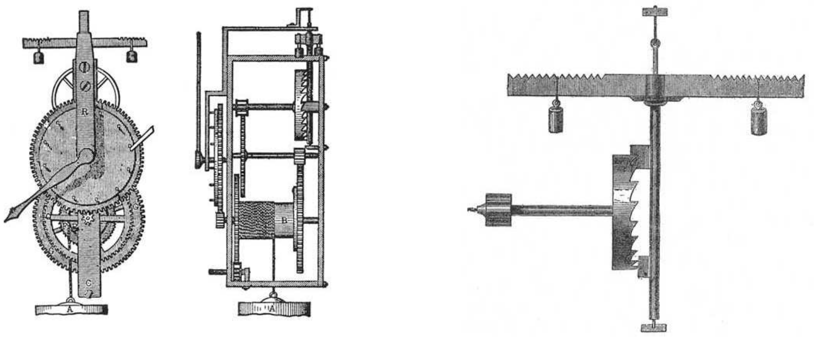
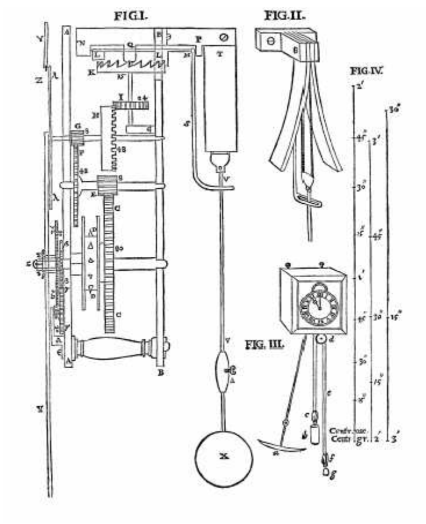
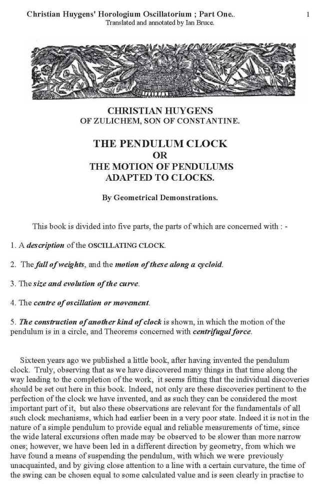


Figure 1. Verge-and-foliot escapement. From "*Encyclopédie, ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers*", edited by Denis Diderot and Jean le Rond d'Alembert in Paris, 1751-1772.





Christian Huygens' Horologium Oscillatorium ; Part One.. 1
Translated and annotated by Ian Bruce.

CHRISTIAN HUYGENS
OF ZULICHEM, SON OF CONSTANTINE.

THE PENDULUM CLOCK
OR
THE MOTION OF PENDULUMS
ADAPTED TO CLOCKS.

By Geometrical Demonstrations.

This book is divided into five parts, the parts of which are concerned with : -

1. A ***description*** of the OSCILLATING CLOCK.

2. The ***fall of weights***, and the ***motion of these along a cycloid***.

3. The ***size and evolution of the curve***.

4. The ***centre of oscillation or movement***.

5. ***The construction of another kind of clock*** is shown, in which the motion of the pendulum is in a circle, and Theorems concerned with ***centrifugal force***.

Sixteen years ago we published a little book, after having invented the pendulum clock. Truly, observing that as we have discovered many things in that time along the way leading to the completion of the work, it seems fitting that the individual discoveries should be set out here in this book. Indeed, not only are these discoveries pertinent to the perfection of the clock we have invented, and as such they can be considered the most important part of it, but also these observations are relevant for the fundamentals of all such clock mechanisms, which had earlier been in a very poor state. Indeed it is not in the nature of a simple pendulum to provide equal and reliable measurements of time, since the wide lateral excursions often made may be observed to be slower than more narrow ones; however, we have been led in a different direction by geometry, from which we have found a means of suspending the pendulum, with which we were previously unacquainted, and by giving close attention to a line with a certain curvature, the time of the swing can be chosen equal to some calculated value and is seen clearly in practise to

Figure 2. Verge and pallet escapement of Huygens' pendulum clock. On the right: title page of Huygens' treatise "*Horologium Oscillatorium sive de motu pendulorum*", 1673, translation by Ian Bruce.

The laws of pendulum were firstly studied by Galileo Galilei between the 16th and 17th centuries and after by Christiaan Huygens in the 17th century. There is an age-old diatribe about the precedence of Galilei or Huygens in realizing the first pendulum clock. Galilei began to reflect upon the pendulum motion in 1581, after observing the oscillations of a lamp suspended inside the Cathedral of Pisa. He had the ingenuous intuition that the oscillation period was independent of the amplitude and found the functional dependence of the pendular period on the suspension length and on the suspended weight. A pendulum could then be used as a tool to measure the time intervals and, for example, could find an application in medicine to measure the pulse rate. Galilei conceived the idea of a "*pulsilogium*" in the last decade of 1500 (Fig. 3) and discussed of it in Padua with his colleague Santorio, who described this medical device in two books of 1620 and 1622. The pendulum length was each time adjusted to synchronize the pulse frequency, permitting thus its calculation. Many years later, in 1641, Galilei proposed the use of the pendulum as a regulatory mechanism for clocks and outlined the related project. However, he was now old and blind and did not accomplish that project. As a matter of fact, it is to be remarked that the ideal pendular motion is strictly isochronous only if the amplitude of its oscillations is very small, as was discovered by Huygens a few decades after the first Galilean studies. Actually, the first pendulum clock was built in 1657 by Huygens, who also conceived the brilliant idea of the cycloidal trajectory, which ensured the theoretical isochronism even for large oscillation amplitudes.

A copy of the original design of Galilei's pendulum clock, which had been traced in those days by Vincenzo Viviani and Vincenzo Galilei, student and son of Galilei respectively, is available to the visitors in the Museum of Galilei in Florence (Fig. 4) and represents the device illustrated by their master in his letter of June 1637 to the Dutch admiral Laurens Reael in order to compete for a prize of 30,000 guldens. In this letter, he described his method for detecting the longitude offshore with the help of the so-called "Jovilabe", by comparing the local time with the hiding periods of Jupiter's Medicean satellites. This comparison depended on the possibility of making an exact measurement of time, and to this end, Galileo proposed the idea of his own pendulum clock. Furthermore, Viviani left also a report on the process that led to the discovery of the pendulum laws and their possible application. Figure 5 shows a reconstruction of the pendulum clock with the Galilei escapement, which was realized in 1879 by the Florentine clock-maker Eustachio Porcellotti on the basis of Viviani's design and is saved in the Museum of Galilei as well. In despite of such previous

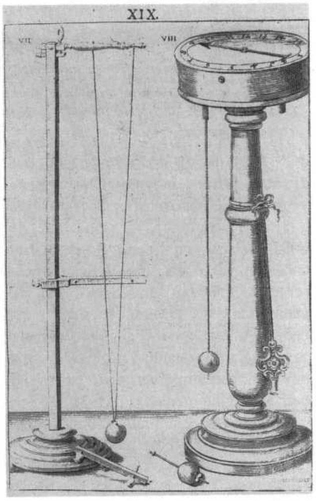


Figure 3. Pendulums used by the *Accademia del Cimento* to measure oscillatory phenomena. The one on the left might be identified with Galilei's *pulsilogium*. From "*Saggi di naturali esperienze fatte nell'Accademia del Cimento*", Florence, 1667.

Figure 4. Copy of the design of Galilei's clock mechanism by Vincenzo Viviani and Vincenzo Galilei. From *Museo Galileo*, Photographic Archives, Florence.

studies by Galilei, the invention of the pendulum clock was claimed in 1658 by Huygens, whose priority was hotly contested by Viviani. It is reported that, observing Viviani's designs, Huygens sharply declared: "It cannot work!".

The regulation by the balance-wheel-coil-spring system was introduced by Hooke afterwards in the late 17th century and in the meanwhile, the escapement evolved from the verge to the anchor, which was introduced by Clement in 1670, and then to the deadbeat, the cylinder and the final lever escapement, which were realized by Graham, Tompion and Mudge respectively in the 18th century and reached a higher precision due to the elimination of any recoil movement. Later on, the clock structure and the working technique had to remain nearly unaltered throughout the modern and contemporary ages until the recent appearance of electric clocks, which however did not caused the disappearance of mechanical clocks.

Pendular Motion Isochronism

The theoretical oscillatory period of the simple pendulum may be calculated by well known procedures ignoring the impulse supply and the energy dissipation in the whole clock mechanism. Defining the swing angle by θ (e. g. positive in the anticlockwise direction), introducing the dimensionless time variable $\tau = \omega_n t$, where $\omega_n = \sqrt{g/l}$ and indicating the derivatives with respect to τ with primes, the motion equation shows the familiar trigonometric law of the restoring force

$$\theta'' + \sin\theta = 0 \tag{1}$$

A first integration gives

$$\frac{\theta'^2}{2} = 2\left(\sin^2\frac{\Theta}{2} - \sin^2\frac{\theta}{2}\right) \tag{2}$$

where Θ is the oscillation amplitude. Hence, putting $\sin^2\Theta/2 = k^2$ and $\sin^2\theta/2 = k^2\sin^2 u$, the change of the variable from θ into u leads to the Legendre normal form, which permits calculating the dimensionless oscillation period T by a second integration

$$d\tau = \frac{du}{\sqrt{1-k^2\sin^2 u}} \qquad \rightarrow \qquad T = 4K(k) = 4K\left(\sin\frac{\Theta}{2}\right) \tag{3a,b}$$

where K stands for the complete elliptic integral of the first kind. This result reveals the dependence of the period on the swing amplitude, because the complete elliptic integral $K(k)$ is an increasing function of the modulus k and tends to $\pi/2$ for $k \to 0$, so that the period increases monotonicly on increasing the amplitude and approaches the harmonic period 2π for small oscillation width.

The cycloidal pendulum described by Huygens in his treatise "*Horologium Oscillatorium sive de motu pendulorum*" in 1673 is not affected by this drawback, because it is based on the tautochronous property of the cycloidal trajectory,

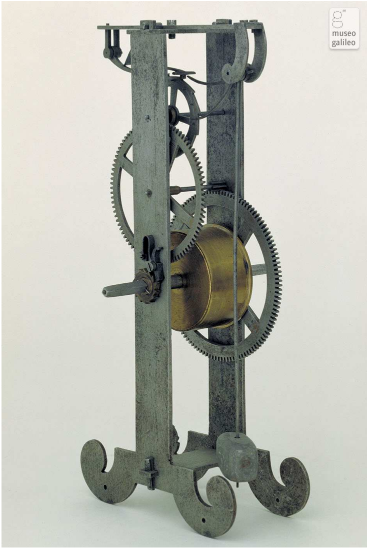


Figure 5. Reconstruction of Galilei's pendulum clock by Eustachio Porcellotti (1879). From *Museo Galileo*, Photographic Archives, Florence.

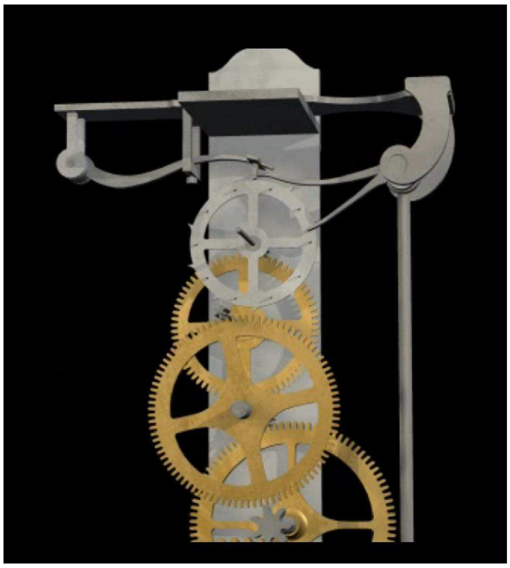


Figure 6. Reconstruction of Galilei's pendular mechanism on the basis of Viviani' design. From *Museo Galileo*, Photographic Archives, Florence.

along which a point mass always slides down in the same lapse of time under the influence of gravity regardless of its starting position.

However, a deeper reflection is here appropriate on the fact that, strictly speaking, all the above considerations on the theoretical isochronism of the pendular motion are somewhat illusory in practice, due to the unavoidable friction losses present in the whole clock assembly and to the consequent impulse supply necessary to provide the dissipated energy periodically.

Figure 6 shows a mechanical reconstruction of Galilei's clock prototype on the basis of Viviani's design and clearly highlights the functionality of the Galilean escapement. The escapement wheel has ten ratchets on the crown and ten front pegs. Its intermittent motion is controlled by a catch, which is slightly loaded by a thin spring, and by two curved levers fastened to the pendulum hinge. On approaching the left dead position of the pendulum near the oscillation end, i. e. for a certain angle $\theta_1 < 0$, the releasing lever raises the catch until the wheel is left free and rotates to contact the lower impulse lever with its peg (pendulum position θ_2). After a more or less short recoil to reach the dead position, the wheel peg pushes the impulse lever until leaving it for $\theta = \theta_3$, near the right dead position, providing the energy lost by friction during the cycle.

Considering the real running of the pendulum machine, Equation (1) changes into

$$\theta'' + \theta = \theta - \sin\theta + \frac{M_{\mathrm{mot.}}(\theta)}{mgl} - \frac{M_{\mathrm{hinge}}\,\mathrm{sgn}(\theta')}{mgl} - \frac{M_{\mathrm{air}}\,\mathrm{sgn}(\theta')}{mgl}\omega_n^{\mu}|\theta'|^{\mu} + \frac{M_{\mathrm{rel.}}(\theta)}{mgl} \tag{4}$$

where m is the pendulum mass and the four moments M are defined as follows.

- $M_{\mathrm{mot.}}$ is the motive torque on the impulse lever, which varies with the pendulum angular tilt, is active from the starting position θ_2 up to the final one θ_3, where the peg leaves the impulse lever, and is proportional to the mutual force between the peg and the lever. This force is in turn proportional to the constant driving torque M_0, applied to the escapement wheel by the motor weight through the whole gear train, and is also a function of θ and of the sliding direction of the peg, which determines the sign of the friction angle.
- M_{hinge} is the absolute value of the resisting friction torque on the pendulum hinge.
- $M_{\mathrm{air}}(\omega_n\theta')^{\mu}$ indicates the air resistance, which is supposed to be a function of the μ^{th} power of the oscillating velocity.
- $M_{\mathrm{rel.}}$ is the moment of the force necessary to release the ratchet, which is supposed active from the position θ_1 to the one θ_2 (it is supposed that the peg contacts the impulse lever immediately after the wheel release).

Assuming one of the above-mentioned angles as a small reference parameter ε, e. g. $\theta_1 = \varepsilon$ and scaling all θ's by ε, the difference $\theta - \sin\theta$ is of order ε^3. If one supposes that all the four dimensionless moments of Eq. (4) exert an influence on the pendulum motion that is comparable with the gravitational non-linearity, they must be regarded of order ε^3 as well. Otherwise, some of them may be regarded of a lower or higher order of magnitude, i. e. of order ε^n with $n \neq 3$.

The right hand of Eq. (4) is characterized by discontinuities of the first kind but, looking only for a first order approximation of the solution, an averaging approach of the Krylov-Bogoliubov type appears appropriate. Putting $\theta = \varepsilon\beta$, letting $\beta = B\sin(\tau + \phi)$ be the zero order solution for $\varepsilon = 0$ and indicating the right hand of Eq. (4) with $\varepsilon^3 f(\beta)$, the K-B procedure assumes that B and ϕ are both functions of τ and eliminates the new degree of freedom that is being introduced by imposing the further condition $\beta' = B\cos(\tau + \phi)$, whence

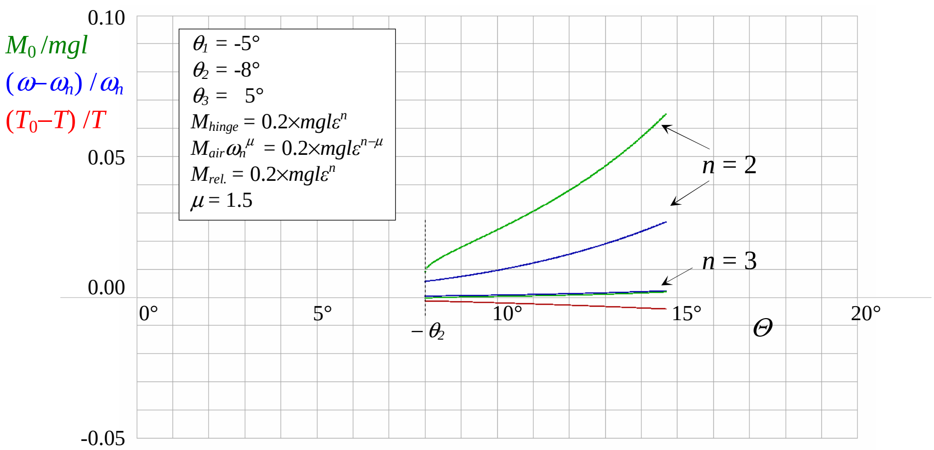


Figure 7. Dimensionless driving torque on escapement wheel M_0/mgl (upper curves, in green) and angular frequency relative change $(\omega-\omega_h)/\omega_h$ (lower curves, in blue) versus oscillation amplitude Θ, for two values of the order n. Non-linear frequency change of ideal pendulum (bottom curve, in red), $T_0 = 2\pi$, $T = 4K(k)$.

$$B'\sin(\tau+\phi)+B\phi'\cos(\tau+\phi)=0 \qquad B'\cos(\tau+\phi)-B\phi'\sin(\tau+\phi)=\varepsilon^2 f(\beta) \tag{5a,b}$$

These equations imply that B and ϕ are very slowly varying functions of τ and can be approximately averaged in the short period 2π neglecting their variation:

$$B'=\frac{\varepsilon^2}{2\pi}\int_0^{2\pi} f(\beta)\cos(\tau+\phi)d(\tau+\phi) \qquad B\phi'=-\frac{\varepsilon^2}{2\pi}\int_0^{2\pi} f(\beta)\sin(\tau+\phi)d(\tau+\phi) \tag{6a,b}$$

Thus, fixing for example the three dimensionless dissipative moments, $M_{\text{hinge}}/(mgl)$, $M_{\text{air}}(\omega_h\varepsilon)^{\mu}/(mgl)$ and $M_{\text{rel.}}/(mgl)$, which are contained in $f(\beta)$, and assuming steady oscillations ($B' = 0$), it is possible to solve for the required driving torque M_0 and the frequency change $\omega_h\phi'$ in dependence on the oscillation amplitude $\Theta = \varepsilon B$.

Figure 7 illustrates these results for an example case and shows the difference between two choices of the orders of magnitude of the four moments $M_{(\ldots)}$ in Eq. (4): $n = 2$ indicates that the non-linear gravitational effect is of a lower order, whereas this effect is comparable with the dissipative and impulsive effects for $n = 3$. The red curve gives the theoretical dependence of the frequency on the amplitude and corresponds to an exponent $n >> 3$. The exponent μ of the air resistance was fixed with the value $\mu = 1.5$, in the implicit hypothesis of an intermediate viscous-turbulent condition. What is most interesting in the results is that the overall effect of the driving impulse and the dissipation sources somehow counterbalances the period increase of the ideal pendulum on increasing the oscillatory amplitude and may even isochronize the motion in particular conditions where the non-linear effects are all comparable with each other.

Conclusive Remarks

An animated debate arose in the seventeenth century between the Dutch scientist Christiaan Huygens and Galilei's heirs, Vincenzo Viviani and Vincenzo Galilei, about the priority of the invention and the construction of the first pendulum clock. What we may conclude now is that both were to be considered fathers of this ingenious instrument: Galilei for having firstly studied the laws of pendular motion and having understood their application in the time measuring; Huygens for his successful finding on the tautochronous property of the cycloidal trajectory, which permits attaining the theoretical pendulum isochronism, and for having really constructed the first pendulum clock.

A scrupulous analysis on the combined effects of the unavoidable dissipation sources present in the clock assembly and of the necessary periodic impulses to be provided in order to restore the lost energy, highlights the slight deviation of the real operation from the ideal theory, whose results appear then somehow illusory, and the possible isochronization of the simple pendulum motion in particular dissipation conditions.

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