A closed-form solution for natural frequencies of thin-walled cylinders with clamped edges

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A B S T R A C T
This paper presents an approximate closed-form solution for the free-vibration problem of thin-walled clamped–clamped cylinders. The used indefinite equations of motion are classic. They derive from Reissner’s version of Love’s theory, properly modified with Donnell’s assumptions, but an innovative approach has been used to find the equations of natural frequencies, based on a solving technique similar to Rayleigh’s method, on the Hamilton’s principle and on a proper constructions of the eigenfunctions.

Thanks to the used approach, given the geometric and mechanical characteristics of the cylinder, the model provides the natural frequencies via a sequence of explicit algebraic equations; no complex numerical resolution, no iterative computation, no convergence analysis is needed.

The predictability of the model was checked both against FEM analysis results and versus experimental and numerical data of literature. These comparisons showed that the maximum error respect to the exact solutions is less than 10% for all the comparable mode shapes and less than 5%, on the safe side, respect to the experimental data for the lowest natural frequency.

There are no other models in the literature which are both accurate and easy to use. The accurate models require complex numerical techniques while the analytical models are not accurate enough. Therefore the advantage of this novel model respect to the others consists in a best balance between simplicity and accuracy; it is an ideal tool for engineers who design such shells structures.

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1. Introduction

Structural elements similar to thin-walled cylinders are widely used in several engineering fields; for example, cylindrical shell-like structures exist in pipelines, submarine hulls, aircraft fuselages and missiles. During mechanical processing needed for their manufacture or during their normal use, these elements are often stressed by time-varying forces; consequently, there is a need to characterize the vibratory behaviour to optimise the design and the production process.

The present paper is composed of five sections and an appendix. This section provides a short historical review of the numerical and analytical models of free vibrations of thin elastic shells. Section 2 presents the differential equations of motion. In Section 3 and in Appendix the mathematical basis of the present model is outlined, and the key equations are derived. A detailed analysis of the results, together with several comparisons with other models and experimental data, is presented in Section 4, followed by conclusions in Section 5.

In the literature, there are several theories with various assumptions and simplifications about the vibrations of thin elastic shells; these theories typically are based on Love’s indefinite equilibrium equations derived at the end of 19th century [1]. The research on this topic intensified during the 1960 s and 1970 s [2] and was further developed in the last two decades [3,4]. Over the years, linear models valid for small deformations were developed, along with non-linear models [5] also valid for large deformations.

In particular, the natural vibrations of thin-walled circular cylindrical shells were extensively analysed both from a theoretical point of view [6–13] and from an experimental point of view [6,7,11]; a recently published study aimed to adapt the classical theories to new applications based on carbon nanotubes [14]. However, due to the complexity of the problem, the exact solution of indefinite equations of motion only exists for circular cylindrical shells with two opposite shear diaphragm edges [12]. With other boundary conditions, the integration of these equations is generally performed with the aid of numerical methods; only in a few cases the solution has been found analytically, thanks to the introduction of special simplifying assumptions, but to the
Applying the Newton form of variables separation for arbitrary boundary conditions and of the frequency equation. Chung [20], using the Sanders’ shell based on coefficients dependent on the constraints of geometry and material characteristics, which can be determined only numerically; therefore, this is not really a closed-form model. Wang and Lai [10] introduced a novel approach based on the wave theory and on the well-known Love’s equations, which allowed them a closed-form resolution for different boundary conditions, clamped–clamped included as in this paper; however, the solution results inaccurate for the simpler mode shapes, as occurred for the Koval and Cranch [7] model. Pellicano [11] conducted both theoretical and experimental analyses on linear and nonlinear vibration based on the Sanders–Koiter theory [22–23] for different boundary conditions; in this case, the analysis was also performed using numerical resolution techniques. Recently, further approaches to the problem were developed: Xing et al. [12], working from the Donnell–Mushtari theory [24], resolved the problem for different boundary conditions via the variables separation method associated with the Newton iterative method; moreover, both Xie et al. [13] and Zhang et al. [25] analysed different boundary conditions using the Goldeneveizer–Novozhilov theory [26] but with different numerical approaches, the former used the Haar wavelet numerical method, while the latter used the local adaptive differential quadrature method. Khalili et al. [27] presented a formulation of 3D refined higher-order shear deformation theory for the free vibration analysis of simply supported–simply supported and clamped–clamped cylindrical shells and the solutions are obtained using the Galerkin numerical method.

The literature review found no models for the free-vibration problem of clamped–clamped cylinders, which are both accurate and easy to use. The accurate models require complex numerical techniques while the analytical models are not accurate enough. The novel model presented here, in contrast, combines good accuracy with ease and speed of calculation: it carefully provides the natural frequencies via a simple sequence of explicit algebraic equations; no complex numerical resolution, no iterative computation, no convergence analysis is needed, unlike other models in the literature or FEM analysis.

The used indefinite motion equations were classic, but an innovative approach was used to find the equations of natural frequencies based on a solving technique similar to Rayleigh’s

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<td>ω</td>
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<td>circular frequency [rad s⁻¹]</td>
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Subscripts

s, r, x

circumferential direction
radial direction
longitudinal direction

Deterioration of accuracy or applicability domain. Arnold and Warburton [6] were among the first to study this type of problem; using the energy method and Timoshenko’s relationships [15], they obtained a closed-form approximation of the natural frequencies for the case of simply supported edges. Koval and Cranch [7] studied the case of clamped–clamped edges using Donnell’s equations [16] and provide an analytical solution as in this paper, but their model gets a limited applicability domain due to several oversimplifications. The same issue was addressed by Smith and Haft [8] using Fliigel’s equations [17] decoupled by Yu [18] but in this case as well, the problem was only solved numerically. Also Xuebin [19] used the Fliigel’s equations but introducing a new form of variables separation for arbitrary boundary conditions and applying the Newton–Raphson iteration method for the resolution of the frequency equation. Chung [20], using the Sanders’ shell equations, obtained the expression of the frequency equation for any kind of boundary condition, but with the aid of iterative numerical method. Callahan and Baruh [9] obtained the natural frequencies analytically for several boundary conditions using the equations of natural frequencies. Also, the calculation was based on coefficients dependent on the constraints of geometry and material characteristics, which can be determined only numerically; therefore, this is not really a closed-form model. Wang and Lai [10] introduced a novel approach based on the wave theory and on the well-known Love’s equations, which allowed them a closed-form resolution for different boundary conditions, clamped–clamped included as in this paper; however, the solution results inaccurate for the simpler mode shapes, as occurred for the Koval and Cranch [7] model. Pellicano [11] conducted both theoretical and experimental analyses on linear and nonlinear vibration based on the Sanders–Koiter theory [22–23] for different boundary conditions; in this case, the analysis was also performed using numerical resolution techniques. Recently, further approaches to the problem were developed: Xing et al. [12], working from the Donnell–Mushtari theory [24], resolved the problem for different boundary conditions via the variables separation method associated with the Newton iterative method; moreover, both Xie et al. [13] and Zhang et al. [25] analysed different boundary conditions using the Goldeneveizer–Novozhilov theory [26] but with different numerical approaches, the former used the Haar wavelet numerical method, while the latter used the local adaptive differential quadrature method. Khalili et al. [27] presented a formulation of 3D refined higher-order shear deformation theory for the free vibration analysis of simply supported–simply supported and clamped–clamped cylindrical shells and the solutions are obtained using the Galerkin numerical method.

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The used indefinite motion equations were classic, but an innovative approach was used to find the equations of natural frequencies based on a solving technique similar to Rayleigh’s
method, on the Hamilton’s principle and on a proper constructions of the eigenfunctions.

The predictability of the model was checked both against FEM modal analysis and versus experimental and numerical data of literature. These comparisons showed that maximum error respect to the exact solutions is less than 10% for all the comparable mode shapes and less than 5%, on the safe side, respect to the experimental data for the lowest natural frequency.

The advantage of this novel model respect to the others consists in a best balance between simplicity and accuracy; therefore it is an ideal tool for engineers who design such shells structures.

2. Differential equations of motion

The indefinite equations of motion used in this paper derive from Reissner’s version [28] of Love’s theory [1] modified with Donnell’s assumptions [16], without the introduction of further simplifications.

Consider a thin-walled circular cylindrical shell, of finite length l, constant thickness h and mean radius a (see Fig. 1) consisting of material having a density $\rho$, Young’s modulus $E$ and Poisson’s ratio $\nu$. Fig. 1 shows the reference surface corresponding to the mean radius and the orthogonal local reference system consisting of longitudinal direction $x$, circumferential direction $s$ and radial direction $r$.

Fig. 2 shows the graphical representation of the forces and moments that arise from the internal stress state and act on the shell. Such internal actions, as well as the inertial forces, are defined per unit of arc length on the reference surface and are considered applied on it.

The forces $N_x$, $Q_x$ and $N_{sx}$ acting on the $x=$ constant face are the components of the vector $F_x$ whereas the forces $N_s$, $Q_s$ and $N_{sx}$ acting on the $s=$ constant face are the components of the vector $F_s$: $F_x = N_x x + N_{sx} s - Q_x r$, $F_s = N_s x + N_{sx} s - Q_s r$.

where $x$, $s$ and $r$ represent the triad of unit vectors of the local reference system. Similarly, the moments $M_x$ and $M_{sx}$ acting on the $x=$ constant face are the components of the vector $M_x$, while the moments $M_s$ and $M_{sx}$ acting on the $s=$ constant face are the components of the vector $M_s$: $M_x = -M_{sx} x + M_s s$, $M_s = -M_{sx} x + M_{sx} s$.

The indefinite equations of motion for a thin-walled circular cylindrical shell can be written as follows [28]:

$$\frac{\partial N_{sx}}{\partial x} + \frac{\partial N_{sx}}{\partial s} = \rho h \frac{\partial^2 u_s}{\partial t^2},$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_x}{\partial s} = \rho h \frac{\partial^2 u_s}{\partial t^2},$$

$$\frac{\partial Q_s}{\partial x} - \frac{\partial Q_s}{\partial s} = \rho h \frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial M_{sx}}{\partial x} + \frac{\partial M_{sx}}{\partial s} - Q_s = 0,$$

$$\frac{\partial M_{sx}}{\partial x} + \frac{\partial M_{sx}}{\partial s} - Q_s = 0.$$

The displacements $u_x$, $u_s$, $u_r$ and the rotations $\beta_x$, $\beta_s$, $\gamma_{sx}$, the curvatures $k_x$, $k_s$ and the torsion $\tau$ are related by the following congruence equations [28]:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} + \frac{u_s}{a} \quad \varepsilon_s = \frac{\partial u_s}{\partial s} + \frac{a}{\partial u_r}{\partial s},$$

$$\beta_x = -\frac{\partial u_r}{\partial x} \quad \beta_s = \frac{u_x}{a} + \frac{\partial u_r}{\partial s},$$

$$k_x = \frac{\partial \beta_x}{\partial s} + \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s} = \frac{\partial \beta_x}{\partial s}.$$
\[
\frac{\partial^2 M_s}{\partial x^2} + \frac{2\partial^2 M_{ss}}{\partial x \partial s} + \frac{\partial^2 M_s}{\partial s^2} - \frac{N_s}{a} = \rho \frac{\partial^2 u_t}{\partial t^2}
\]

where the third equation was obtained by combining the last equations of (1). The second assumption concerns the congruence Eq. (2.3); it is assumed that variation in the circumferential displacement \( u_s \) does not influence the curvature \( k_s \) and the torsion \( \tau \); Eq. (2.3) are then reduced to:

\[
k_s = \frac{\partial^2 u_s}{\partial x^2}, \quad k_s = -\frac{\partial^2 u_t}{\partial s^2}, \quad \tau = -2\frac{\partial^2 u_t}{\partial x \partial s}
\]

These latter Eq. (5) are equal to the corresponding equations of the plate theory [15]. The two assumptions explained above are equivalent to stating that a thin walled cylinder with relatively small curvature behaves similarly to a thin plate; the only differences, due to the curvature of the wall, are the presence of \( N_s/a \) in the third equation of motion and the presence of \( u_t/a \) in the equation for normal deformation in the circumferential direction \( \varepsilon_s \). Finally, the constitutive equations are as follows [28]:

\[
\begin{align*}
N_s &= K(e_s + \nu e_s), \quad N_t = K(e_t + \nu e_t), \\
N_{ss} &= N_{st} = \frac{1}{2}(1 + \nu)\varepsilon_{ss}, \\
M_s &= D(k_s + \nu k_s), \quad M_t = D(k_t + \nu k_t), \\
M_{ss} &= M_{st} = \frac{1}{2}(1 + \nu)\varepsilon_{ss} = G\theta^2 \tau/12
\end{align*}
\]

where \( G \) is the shear modulus and

\[
K = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{12(1 - \nu^2)}
\]

are, respectively, the extensional and bending rigidity of the shell wall.

Substituting the congruence Eqs. (2.1), (2.2) and (5) into the constitutive Eq. (6) and subsequently replacing into the indefinite equations of motion (4), the latter are expressed as functions of the displacements:

\[
K \left[ \frac{\partial^2 u_s}{\partial x^2} + \frac{1 - \nu}{2a^2} \frac{\partial^2 u_t}{\partial y^2} + \frac{1 + \nu}{2a^2} \frac{\partial^2 u_t}{\partial x^2} + \frac{\nu}{a} \frac{\partial u_t}{\partial x} \right] = \rho \frac{\partial^2 u_t}{\partial t^2},
\]

\[
K \left[ \frac{1 + \nu}{2a^2} \frac{\partial^2 u_t}{\partial x^2} + \frac{1 - \nu}{2a^2} \frac{\partial^2 u_t}{\partial y^2} + \frac{\nu}{a} \frac{\partial u_t}{\partial x} + \frac{1}{a} \frac{\partial u_t}{\partial x} \right] = \rho \frac{\partial^2 u_t}{\partial t^2}
\]

3. Equation of natural frequencies

A procedure similar to Rayleigh’s method was used to determine the natural frequencies of the system. Based on the considerations presented below, three displacement functions \( u_s, u_r, u_t \) were defined and adopted as the eigenfunctions of the problem. However, the corresponding eigenvalues were identified by making Hamilton’s action constant instead of making Rayleigh’s quotient constant. In fact, Hamilton’s principle declares that the natural motions of a mechanical system in which the extreme configurations are known are those that make the Hamiltonian action \( H \) constant for each possible configuration. For a conservative system, this principle mathematically becomes:

\[
\delta H = \delta \int_{t_0}^{t_1} L dt = 0
\]

or

\[
\int_{t_0}^{t_1} \delta W dt = 0
\]

where \( L \) is the Lagrangian function (difference between the kinetic and potential energy of the system), \( t_0 \) and \( t_1 \) are the generic temporal limits of integration and \( \delta W \) is the virtual work performed by all forces present in the system, including the inertial forces.

The second formulation (9.2) allows one to write Hamilton’s principle without deriving the Lagrangian function \( L \) when the equations of motion are already known. In this case, the virtual work \( \delta W \) can be easily expressed by multiplying each of the undefined scalar equations of motion for a virtual reversible displacement along the corresponding direction and so, using Eq. (7):
The deformation in free vibration of a thin-walled circular cylinder consists of a positive integer $n$ of waves on the section orthogonal to the axis and a positive integer $m$ of half-waves on the section containing the axis, henceforth called circumferential waves and longitudinal half-waves, respectively. Therefore, each mode shape is characterized by a pair of values of $n$ and $m$ (see, for example, Fig. 4).

The shape of the circumferential waves is independent of the boundary conditions, while the shape of the longitudinal half-waves depends on boundary conditions and is similar to the flexural vibrations of beams subject to the same constraints [10,13].

These considerations, together with the need to construct eigenfunctions that, respecting the constraint conditions and mutual orthogonality, were analytically manageable in the subsequent steps of derivation and integration, led the authors to distinguish between odd and even numbers of longitudinal half-waves (see Appendix for more details). For odd $m$:

$$u_x = A_n \left[ -\sin \mu \left( \frac{1}{2} - x \right) + \nu \sinh \mu \left( \frac{1}{2} - x \right) \right] \cos \theta \cos \omega t,$$

$$u_t = A_n \left[ \cos \mu \left( \frac{1}{2} - x \right) + \nu \cosh \mu \left( \frac{1}{2} - x \right) \right] \sin \theta \cos \omega t,$$

$$u_r = A_n \left[ \cos \mu \left( \frac{1}{2} - x \right) - \nu \cosh \mu \left( \frac{1}{2} - x \right) \right] \cos \theta \cos \omega t,$$

(11)

where $A_n, A_t$, and $A_r$ are arbitrary coefficients, $\omega$ is the circular frequency, $X=x/l$ and, to comply with the boundary conditions,

$$\nu = \frac{\sin (\mu/2)}{\sinh (\mu/2)}.$$

(12)

The quantity $\mu$ must satisfy the equation

$$\tan \frac{\mu}{2} + \tanh \frac{\mu}{2} = 0,$$

(13)

whose roots are (in addition to the value 0):

$$\mu \approx \{1.506 + (m - 1)\pi\},$$

(14)

for $m=1, 3, 5, 7$.

For even $m$, the eigenfunctions become:

$$u_x = A_n \left[ -\cos \mu \left( \frac{1}{2} - x \right) + \nu \cosh \mu \left( \frac{1}{2} - x \right) \right] \cos \theta \cos \omega t,$$

$$u_t = A_n \left[ \sin \mu \left( \frac{1}{2} - x \right) - \nu \sinh \mu \left( \frac{1}{2} - x \right) \right] \sin \theta \cos \omega t,$$

$$u_r = A_n \left[ \sin \mu \left( \frac{1}{2} - x \right) - \nu \sinh \mu \left( \frac{1}{2} - x \right) \right] \cos \theta \cos \omega t,$$

(15)

in which the quantity $\mu$ satisfies the equation

$$\tan \frac{\mu}{2} - \tanh \frac{\mu}{2} = 0,$$

(16)

whose roots are (in addition to the value 0):

$$\mu \approx \{2.500 + (m - 2)\pi\},$$

(17)

for $m=2, 4, 6, 8, \ldots$

and $\psi$ is again Eq. (12).

Substituting into Eq. (10) both the partial derivatives $\partial \ldots / \partial \omega$... of $u_x, u_t$, and $u_r$, and integrating with respect to $x$ and $\theta$ gives the elementary virtual work. Subsequently, integrating Eq. (9.2) with respect to time $t$, after suitable manipulation, gives:

$$\left[ \left( \frac{E}{d^2} \eta_1 + \frac{1}{2} (1 - \nu) n^2 \eta_2 - \eta_1 \Delta \right) A_n - \frac{1}{2} \eta \left( \frac{1}{2} + 2 \xi \eta \eta_2 \eta_1 - \xi \eta_2 A_1 - \xi \eta_2 A_{12} \right) \right] \delta A_x + \left[ \frac{1}{2} \eta \eta_2 \eta_1 - \eta_1 \eta_2 A_1 - \frac{h^2}{12 \pi^2} \left( \frac{E}{d^2} \eta_1 + n^2 \eta_1 + 2 \xi \eta \eta_2 \eta_1 \right) A_1 \right] \delta A_1 = 0,$$

(18)

where

$$\Delta = \rho a^2 (1 - \nu^2) \frac{\partial^2 \omega^2}{\pi^2},$$

(19)

is the dimensionless frequency factor and

$$\xi = \frac{\mu a}{\pi}.$$

(20)

Moreover, for an odd number $m$ of longitudinal half-waves

$$\eta_1 = 1 + \eta^2, \quad \eta_2 = 1 - \eta^2 + \frac{2}{\mu} \sin \mu,$$

(21)

while for an even number $m$ of longitudinal half-waves,

$$\eta_1 = 1 + \eta^2, \quad \eta_2 = 1 - \eta^2 + \frac{2}{\mu} \sin \mu.$$

(22)

Because the quantities $\delta A_x$, $\delta A_1$, and $\delta A_2$ are arbitrary, Eq. (18) can be satisfied only if the quantities in the braces vanish individually. If these quantities are set to zero, a homogeneous system of three linear equations in three unknowns is obtained: $A_n, A_t$, and $A_r$. To avoid the trivial solution, the determinant of the coefficients is set to zero, giving the following cubic equation for the frequency factor $\Delta$:

$$\Delta^3 - R_2 \Delta^2 + R_1 \Delta - R_0 = 0,$$

(23)

where

$$R_2 = \left( \frac{n^2}{\eta_1} + \frac{4 \eta_2}{\eta_1} \right) \frac{E}{d^2} + 1 + 2 \xi \eta \eta_2 + \frac{2 \xi}{\eta_1} \left( \xi^2 + n^2 + 2 \xi \eta \eta_2 \right),$$

$$R_1 = \frac{1}{2} \left( \eta_1 + \eta_2 \right) \frac{E}{d^2} + \frac{4 \xi}{\eta_1} \eta \eta_2 \eta_1 + \frac{2 \eta}{\eta_1} \left( \frac{1 - \nu - 2 \xi}{\eta} \right),$$

$$R_0 = \frac{1}{2} \left( \eta_1 + \eta_2 \right) \frac{E}{d^2} + \frac{4 \xi}{\eta_1} \eta \eta_2 \eta_1 + \frac{2 \eta}{\eta_1} \left( \frac{1 - \nu - 2 \xi}{\eta} \right).$$

(24)
\[ R_0 = \frac{1 - \nu}{\eta_1} \left[ \frac{1 + \nu\eta_2 - \eta_1}{\eta_1} \right]^2 \frac{\psi^4 + \frac{h^2}{12\alpha^2} \left( \frac{\psi^2 + n^2}{\eta_1} \right)^2}{\left( \psi^2 + n^2 + 4\psi^2n^2 \right) \eta_1^2}. \]

Eq. (23) has three different, real and positive roots:

\[ \Delta_1 = 2a^{1/3} \cos \frac{\varphi + 2\pi}{3} + \frac{R_0}{3} \]
\[ \Delta_2 = 2a^{1/3} \cos \frac{\varphi + 4\pi}{3} + \frac{R_0}{3} \]
\[ \Delta_3 = 2a^{1/3} \cos \frac{\varphi + 2\pi}{3} + \frac{R_0}{3}. \]

(25)

where

\[ \alpha = \left[ \frac{1}{27} \left( R_1 - R_2^2 \right)^{3/2} \right] \cdot \varphi = \frac{\cos \left[ \frac{1}{2a} \left( R_0 - R_1 R_2^3 + 2R_3^3 \right) \right]}{2a^2 \rho (1 - \nu^2)}. \]

(26)

Once the values of \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) were known, the three natural frequencies \( f_1, f_2 \) and \( f_3 \) were calculated by simple manipulation of Eq. (19):

\[ f_{1,2,3} = \omega_{1,2,3} \frac{a^2}{2\pi} = \frac{1}{2\pi a^2} \frac{E\Delta_{1,2,3}}{\rho (1 - \nu^2)}. \]

(27)

Despite the complex mathematics used to obtain the natural frequencies (27), the practical use of the model is much simpler; given the geometric and mechanical characteristics of the cylinder (length \( l \), mean radius \( a \), thickness \( t \), Young's modulus \( E \), Poisson's ratio \( \nu \) and density \( \rho \)), for fixed mode shape (values \( m \) and \( n \)) the following parameters are sequentially calculated using algebraic equations: \( \mu \) via (14) or (17), \( \eta_1 \) via (12), \( \eta_2 \) via (21) or (22), \( R_0, R_1 \) and \( R_2 \) via (24), \( \alpha \) and \( \varphi \) via (26), \( \Delta \) via (25) and finally \( \eta \) via (27).

The system of linear equations that relates \( A_0, A_1 \) and \( A_2 \) is homogeneous, so the following amplitude ratios can be obtained:

\[ A_0 = \frac{\frac{1}{2} \nu \eta_2 + \frac{1}{2} (1 + \nu) \eta_1 n^2 \eta_1}{\eta_1 - \eta_0} + \frac{1}{2} \nu \eta_2 + \frac{1}{2} (1 + \nu) \eta_1 n^2 \eta_1} \]
\[ A_1 = \frac{\frac{1}{2} \nu \eta_2 + \frac{1}{2} (1 + \nu) \eta_1 n^2 \eta_1 - \Delta \eta_1}{\eta_1 - \eta_0} + \frac{1}{2} \nu \eta_2 + \frac{1}{2} (1 + \nu) \eta_1 n^2 \eta_1 - \Delta \eta_1} \]

(28)

Therefor, for each mode shape (defined by a pair of values \( n \) and \( m \)), the model analytically provides three natural frequencies and two amplitude ratios. These results are in close correspondence with analogous studies [6,10,29]. Each vibration mode is characterized by one natural frequency and one mode shape. The essential difference between a vibration mode and another with the same mode shape consists, in addition to the frequency, in the

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relative amplitude of the displacements $u_x$, $u_r$, and $u_s$, as showed in the following section.

Table 2

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<th>A_x/A_s</th>
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Table 1 shows the frequencies $f_1$, $f_2$ and $f_3$ for a cylinder having the same geometric, physical and mechanical characteristics of those used by other authors [7,8,12]. It is interesting to note that the $f_1$ frequencies are smaller, by one or two orders of magnitude, than frequencies $f_2$ and $f_3$. These results are in keeping with previous studies [7,8,12]. Furthermore, $f_2$ and $f_3$ increase monotonically with an increase in the number of $n$ and $m$, which is in line with the results from the free vibrations of beams and plates, where the natural frequencies increase with the complexity of the waveforms. On the contrary, for fixed $m$ and variable $n$, the $f_1$ frequencies show a minimum. The value of $n$ for which $f_1$ is minimum grows as $m$ increases. Additionally, at fixed $n$, the $f_1$ frequencies increase monotonically with $m$ if $n < 12$, while showing a minimum for $n \geq 12$. The seemingly anomalous trend of the $f_1$ frequencies, which first decrease and then increase with $n$, was first observed by Arnold and Warburton [6] who, in the case of freely supported ends, were able to explain the phenomenon.

4. Results and discussion

Fig. 5. Comparison between the $f_1$ natural frequencies calculated with the present model and other results, for $m \leq 4$ and $n \leq 14$ ($a=3 \text{ in } \approx 76 \text{ mm}, l=12 \text{ in } \approx 305 \text{ mm}, h=0.01 \text{ in } \approx 0.254 \text{ mm}, \rho=0.283 \text{ lb/in}^2 \approx 7833 \text{ kg/m}^2, E=30\times10^6 \text{ psi } \approx 207 \text{ kN/mm}^2, \nu=0.3$).
by considering the strain energy associated with bending and stretching of the reference surface.

Table 2 shows an example of the amplitude ratios (28) calculated for $m \leq 10$ and for $n = 6$. Similar trends are obtained for other values of $n$. These results indicate that, at the lowest natural $f_i$ frequency, the predominant amplitude is $A_r$ and the motion associated with this frequency is, therefore, mostly radial. This mode of free vibration is then called transverse mode. Conversely, associated with this frequency is, therefore, mostly radial. This mode of free vibration associated with $f_1$, therefore, is mostly radial. This mode of free vibration associated with $f_1$ is usually referred to as longitudinal mode and that associated with $f_2$ is called circumferential mode.

In the literature [6–13], particular attention was devoted to the study of the $f_1$ frequencies because of their greater importance to the resonance problems.

To check the validity of the model presented in this study, the calculated $f_1$ frequencies have been compared as a first step with the results of FEM modal analysis conducted by the authors. The FEM analysis was realized in ANSYS 14 using 5856 SHELL181 linear elements. This level of discretization was chosen after a convergence analysis that allowed the authors to assess the modal convergence. On the contrary the two numerical methods seem indistinguishable from one another. On the contrary the two analytical models part from other trends and become inaccurate for small values of $n$, as the same authors of the two models admit.

In light of the above, in the next steps, in order to quantify the percentage differences between the various results but also to reduce the number of possible comparisons only the most relevant results were taken into account, i.e. the mode shapes with $n \leq 8$. Moreover only the Wang and Lai analytical model and the Xing et al. numerical model were considered. The first, in effect, proved to be more accurate than Koval and Cranch model, while the second has been chosen since it uses the same infinite equations of motion of the present paper but, using numerical methods of resolution, gets the "exact" solutions. This induced the authors to consider these latter results, together with the experimental data of Koval and Cranch, as benchmark for assessing the accuracy of the present model. Tables 3 and 4 report these further outcomes. They show that the maximum error of the present model respect to the experimental data is less than 17% (for $m = 1, n = 3$) while the discrepancy respect to Xing et al. exact solutions is within a maximum of 10% (for $m = 3, n = 8$). However, it is worth noting that the maximum difference between the Xing et al. exact solutions and the experimental data is at 13% (for $m = 1, n = 3$). On the other

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Table 3
Comparison of $f_i$ frequencies [Hz] from present model with other alternative studies for $m \leq 4$ and $n \leq 8$, ($a = 3\text{ in} \approx 76\text{ mm}$, $l = 12\approx 305\text{ mm}$, $h = 0.01\approx 0.254\text{ mm}$, $\rho = 0.283 \text{ lb/in}^3 \approx 7833 \text{ kg/m}^3$, $E = 30\times 10^6\text{ psi} \approx 207\text{ kN/mm}^2$, $\nu = 0.3$), n.a. = not available.
hand, it is also to be considered that the disagreement between theory and experiments could be partly due to the imperfect clamping of the experimental specimens as well as to the unavoidable measurement error.

As regards the Wang and Lai analytical model, the maximum error is at 33% (for \( m=1, n=3 \)) respect to the experimental data and at 41% (for \( m=1, n=1 \)) versus Xing et al.

Finally, with reference to the lowest experimental natural frequency, properly identified by all models for \( m=1 \) and \( n=6 \), the present model gives an underestimation of 4.6%, Xing et al. an overestimation of 2.5% and Wang and Lai an overestimation of 5.2%.

From the analysis of the above results, it can be concluded that the presented model is one of better accuracy for the mode shapes, also for those more difficult to identify, and for the case of rotating shells. Presence of Coriolis and centrifugal accelerations as well as the hoop tension due to angular velocities in the differential equations of motion, with the related complexity of implementation and of solution convergence. The analytical model has ease of calculation comparable to the present model but fail for a small value of circumferential waves.

A comparative analysis with experimental and numerical data from the literature showed that the maximum error respect to the exact solutions is less than 10% for all the comparable mode shapes and less than 5%, on the safe side, respect to the experimental data for the lowest natural frequency.

Therefore the advantage of the proposed model respect to the others consists in a better balance between simplicity and accuracy, resulting an ideal tool for engineers who design such shells structures.

Extensions of the present approach for different boundary conditions and for the case of rotating shells are under consideration. The real constraints are yielding, and the use of proper suspension systems could have a beneficial effect on rotor hysteretic instability [30,31]. Furthermore, when shells of revolutions rotate, it is necessary to take into account the Coriolis and centrifugal accelerations as well as the hoop tension due to angular velocities in the differential equations of motion. These effects have significant influence on the dynamic behaviour of rotating shells, and their structural frequency characteristics are qualitatively altered [32–34].

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Appendix

In general, the displacement of a point in a thin shell is a function of the position and of the time, i.e.

\[ u = u(x, \theta, t) \]

however, in order to find approximate solutions of the indefinite equations of motion, analogously to other problems regarding vibrations of continuous systems, it is convenient to write each eigenfunction in the form

\[ u = A \cdot f(x) \cdot g_\theta(\theta) \cdot \cos \omega t \]

while the mutual orthogonality conditions require:

\[ \int u_x u_x dV = 0 \]
\[ \int u_t u_t dV = 0 \]

where \( V \) is the volume of the cylinder, and so

\[ \int_0^l f_i(x) f_j(x) dx \int_0^{2\pi} g_i(\theta) g_j(\theta) d\theta = 0 \]
\[ \int_0^l f_i(x) f_j(x) dx \int_0^{2\pi} g_i(\theta) g_j(\theta) d\theta = 0 \]

Considering that the shape of the circumferential waves is independent of the boundary conditions while the shape of the longitudinal half-waves depends on boundary conditions and is similar to the flexural vibrations of \( g_i(\theta) = \cos(n\theta) \) beams subject to the same constraints [10,13], for the radial displacement \( u_r \), it is suitable to chose and \( f(x) \) similar to the eigenfunctions of the beam subject to the same constraints. As regards the functions \( g_\theta \) and \( f(x) \) of the other displacements, \( u_t \) and \( u_r \), it easy to see that both the boundary conditions and the mutual orthogonality conditions result identically satisfied if one puts

\[ f_i(x) = \frac{d}{dx} f_i(x) \]
\[ g_i(\theta) = g_i(\theta) \]
\[ f_i(x) \propto f_i(x) \]
\[ g_i(\theta) = \frac{d}{d\theta} g_i(\theta) \]

and therefore:

\[ u_x = A \cdot \frac{d}{dx} f_i(x) \cdot \cos(n\theta) \cdot \cos \omega t \]
\[ u_t = A \cdot f_i(x) \cdot \sin(n\theta) \cdot \cos \omega t \]
\[ u_r = A \cdot f_i(x) \cdot \cos(n\theta) \cdot \cos \omega t \]

In this way, all the functions [A.3] and [A.4] are proportional to \( f_i \) or \( \frac{d}{dx} f_i \) and, being \( f_i \), the eigenfunction of the clamped-clamped beam, \( f_i = 0 \) and \( \frac{d}{dx} f_i = 0 \) at either end, and consequently the boundary conditions [A.1] result satisfied. Moreover, the mutual orthogonality conditions [A.2] reduce to

\[ \int_0^l f_i(x) f_j(x) \int_0^{2\pi} [g_i(\theta)]^2 d\theta = 0 \]
\[ \int_0^l f_i(x) dx \int_0^{2\pi} g_i d\theta = 0 \]

which are identically satisfied because the upper and the lower integration limits in the first or in the second integral of Eq. (A.5) are equal (remember that \( g_i(\theta) = \cos(n\theta) \)).

As regards the most convenient form to be given to the \( f_i(x) \) function, consider the following. As well known, the mode shapes of the clamped-clamped beam are proportional to:

\[ f_i(x) = (\sin \mu X - \sin \mu x) + \Psi(\cos \mu X - \cosh \mu X) \]

where \( X = x/l \), \( \Psi = \sin \mu x - \sinh \mu x \), and \( \mu \) indicates one of the infinite roots of the frequency equation \( \cos \mu \cos \mu = 1 \).

Therefore, substituting into Eq. (10) both the partial derivatives \( \partial \ldots \) and the virtual displacements \( \ldots \) of \( u_r \), \( u_t \) and \( u_r \), we will have 80 addends in the first line, 80 in the second, and 112 in the third, for a total of 272 initial addends to be collected and then integrated. However, the symmetry of the boundary conditions may yield a simplified expression for \( f_i(x) \) with only two addends provided that the symmetric and anti-symmetric waves are considered separately. In this way, we have 20 addends in the first line, 20 in the second, and 28 in the third, for a total of 68 initial addends to be collected and integrated.

Then, for the odd numbers \( m \) of the longitudinal half-waves, the eigenfunctions can be written as:

\[ f_i(x) = \cos \mu \left( \frac{1}{2} x \right) + \Psi \cosh \mu \left( \frac{1}{2} X \right) \]

where to comply with the boundary conditions \( \Psi = \frac{\sin \mu X}{\sinh \mu X} \) and \( \mu \) must satisfy the equation

\[ \tan \frac{n}{2} + \tanh \frac{\mu}{2} = 0 \]

For the even numbers \( m \) of the longitudinal half-waves Eq. (A.6) must be modified by replacing \( \cos \leftrightarrow \sin \), \( \cosh \leftrightarrow -\sinh \), so

\[ f_i(x) = \sin \mu \left( \frac{1}{2} x \right) - \Psi \sinh \mu \left( \frac{1}{2} X \right) \]

in which \( \Psi \) is the same as before but \( \mu \), this time, must satisfy the equation

\[ \tan \frac{n}{2} + \tanh \frac{\mu}{2} = 0 \]

Therefore, finally, replacing Eqs. (A.6) or (A.7) into Eq. (A.3) one gets Eqs. (11) and (15) respectively.

References

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